Remark. *All sets are assumed to be topological spaces and all maps are assumed to be continuous unless stated otherwise. The symbol '=' will denote that two topological spaces are homeomorphic. The closed unit interval will be denoted by I or J.*

Exercise 1. *Prove that being homotopic is an equivalence relation (on the set of continuous maps between topological spaces).*

Solution. Let $f, g, k : X \to Y$ be such that $f \sim g$, $g \sim k$, i.e. there exist maps $h, h' : X \times I \to Y$ such that $h(x,0) = f(x)$, $h(x,1) = g(x)$, $h'(x,0) = g(x)$, $h'(x,1) = k(x)$.

- Reflexivity: the map $H_1 : X \times I \to Y$ defined by $H_1(x,t) := f(x)$ for all $t \in I$ is a homotopy between f and itself.
- Symmetry: the map $H_2 : X \times I \to Y$ defined by $H_2(x,t) := h(x, 1-t)$ for all $t \in I$ is a homotopy between g and f.
- Transitivity: the map $H_3 : X \times I \to Y$ defined by $H_3(x,t) := \begin{cases} h(x,2t) & \text{for } 0 \leq t \leq \frac{1}{2} \\ h'(x,2t) & \text{for } \frac{1}{2} < t \leq 1 \end{cases}$ is a homotopy between f and k.

□

Exercise 2. *Let $\simeq$ be an equivalence relation on a topological space X. Prove that the map $f : X/\simeq \to Y$ is continuous iff $f \circ p : X \to Y$ is continuous, where $p : X \to X/\simeq$ is the canonical quotient projection.*

Solution. The direction "⇒" follows from the facts that p is continuous (in fact, the quotient topology is the final topology with respect to p) and the composition of continuous functions is again continuous. For "⇐", let $U \subseteq Y$ be open. Then

$$p^{-1}(f^{-1}(U)) = (f \circ p)^{-1}(U)$$

is open by continuity of $f \circ p$, so $f^{-1}(U)$ must also be open by the definition of quotient topology and we are done. □

Exercise 3. *Show that $D^n/S^{n-1} = S^n$ using the map $f : D^n \to S^n$ given by*

$$f(x_1, \ldots, x_n) = (2\sqrt{1 - \|\boldsymbol{x}\|}\boldsymbol{x}, 2\|\boldsymbol{x}\|^2 - 1).$$

Solution. It's easy to see that f is continuous. Moreover, its restriction to the interior of D^n gives a bijection to $S^n \setminus \{(0, \ldots, 0, 1)\}$ (the inverse function is given by $(\mathbf{y}, z) \mapsto \frac{1}{\sqrt{\frac{1-z}{2}}}\mathbf{y}$) and we have $f(S^{n-1}) = \{(0, \ldots, 0, 1)\}$, so we can define $f' : D^n/S^{n-1} \to S^n$ by $f'([\mathbf{x}]) = f(\mathbf{x})$. Then f' is a bijection, and by the previous exercise it is continuous. Finally, both D^n/S^{n-1} and S^n are compact (Hausdorff) spaces (since both D^n and S^n are closed bounded subsets of $\mathbb{R}^n$ and R^{n+1}, respectively, and $S^{n-1} \subseteq D^n$ is closed), so f' must be a homeomorphism (a general fact for continuous bijections between compact spaces). □

Exercise 4. *Let $f : X \to Y$ and $M_f = X \times I \cup_{j\times 1} Y$. Moreover, let $\iota_X : X \to M_f$ be given by $x \mapsto (x, 0)$, $\iota_Y : Y \to M_f$ be given by $y \mapsto [y]$ and $r : M_f \to Y$ be given by $r(y) = y$, $r(x, t) = f(x)$. Show that*

i) Y is a deformation retract of M_f,

ii) $r \circ \iota_X = f$,

iii) $\iota_Y \circ f \sim \iota_X$.

Solution.

i) Geometrically, the deformation retraction is realized by pushing X along I towards Y.

ii) We have $r \circ \iota_X(x) = r(x, 0) = f(x)$ for all $x \in X$.

iii) The required homotopy $h : X \times J \to M_f$ is given by $h(x, s) = [(x, s)]$.

□

Exercise 5. *Show that the pair (M_f, X) has the homotopic extension property (HEP), i.e. ι_X is a cofibration.*

Solution. Let $g : I \times J \to \{0\} \times J \cup I \times \{0\}$ be any retraction such that $g(0, s) = (0, s)$ and $g(1, s) = (1, 0)$. Then the map $r : M_f \times J \to X \times \{0\} \times J \cup M_f \times \{0\}$ defined by $r(x, t, s) = (x, g(t, s))$ and $r(y, s) = (y, 0)$ is the required retraction. □

Exercise 6. *The smash product between two based spaces is defined by*

$$(C, c_0) \wedge (D, d_0) := (C \times D)/ (C \times \{d_0\} \cup \{c_0\} \times D)\,.$$

Show that $X/A \wedge Y/B = (X \times Y)/(X \times B \cup A \times Y)$.

Solution. Let $p_1 : X \times Y \to X/A \times Y/B$ be given by $p_1(x, y) = ([x], [y])$ and $p_2 : X/A \times Y/B \to X/A \wedge Y/B$ be given by $p_2(([x], [y])) = ([[x]], [[y]])$. Then the composition $p_2 \circ p_1$ is continuous and factors through $(X \times Y)/(X \times B \cup A \times Y)$, which implies that the canonical bijection between $(X \times Y)/(X \times B \cup A \times Y)$ and $X/A \wedge Y/B$ is continuous (using exercise 2). Using the definition of quotient topology several times, it can be shown that this bijection is also open, hence a homeomorphism. □

Exercise 7. *Let $A = \{\frac{1}{n} \cup \{0\}\} \subseteq \mathbb{R}$. Show that (I, A) does not have the HEP, i.e. the inclusion $A \hookrightarrow I$ is not a cofibration.*

Solution. If $A \times J \cup I \times \{0\}$ was a retract of $I \times J$, the retraction would have to preserve connected subsets. But $A \times J \cup I \times \{0\}$ is not locally connected while $I \times J$ is, a contradiction.

□

Exercise 8. *Show that* $(S^m, *) \wedge (S^n, *) \cong (S^{m+n}, *)$.

Solution. Using exercises 3 and 6 from the previous tutorial, we have

$$\begin{aligned}
(S^m, *) \wedge (S^n, *) &\cong (D^m/S^{m-1}) \wedge (D^n/S^{n-1}) \cong \\
&\cong (D^m \times D^n)/(S^{m-1} \times D^m \cup D^n \times S^{n-1}) \cong \\
&\cong (I^m \times I^n)/(\partial I^m \times I^n \cup \partial I^n \times I^m) = \\
&= I^{m+n}/(\partial(I^{m+n})) \cong D^{m+n}/\partial D^{m+n} \cong S^{m+n}.
\end{aligned}$$

□