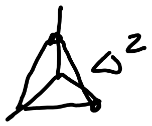
Exercise 1. *Show $\partial\partial = 0$. Use formula $\varepsilon^i_{n+1} \circ \varepsilon^j_n = \varepsilon^{j+1}_{n+1} \circ \varepsilon^i_n$, where $i \leqslant j$. The definition for $\sigma \in C_n(X)$, $\sigma\colon \Delta^n \to X$, is*

$$\partial\sigma = \sum_{i=0}^{n} (-1)^i \sigma \circ \varepsilon^i_n.$$



$C_n(X)$ volná abelovská grupa generovaná singulárními n-simplex $\sigma : \Delta^n \to X$

$\Delta^n = \{(t_0, \dots t_n) \in \mathbb{R}^{n+1}; \sum_0^n t_i = 1\}$

Hraniční operátor $\partial\sigma = \sum_{i=0}^{n} (-1)^i \sigma \circ \varepsilon^i_n$

$$\varepsilon^i_n : \Delta^{n-1} \to \Delta^n = \{(t_0, \dots t_n) ; \sum t_i = 1\}$$

$$\varepsilon^i_n (t_0, t_1, \dots t_{n-1}) = (t_0, t_1, \dots, t_{i-1}, \underset{\overset{\uparrow}{i}}{0}, t_i, \dots t_{n-1})$$

Nechť $i < j$ (případ $i = j$ prověřit zvlášť)

$$L = \varepsilon^i_{n+1} \circ \varepsilon^j_n (t_0, \dots, t_{n-1}) = \varepsilon^i_{n+1} (\underbrace{t_0, \dots, t_{j-1}}_{\text{zde je } i}, 0, t_j, \dots t_{n-1})$$

$$= (t_0, \dots, t_{i-1}, 0, t_i, \dots t_{j-1}, 0, t_j, \dots t_{n-1})$$

$$P = \varepsilon^{j+1}_{n+1} \circ \varepsilon^i_n (t_0, \dots, t_{n-1}) = \varepsilon^{j+1}_{n+1} (t_0, \dots, t_{i-1}, 0, t_i, \underbrace{\dots t_{n-1}}_{\text{zde je } j})$$

$$= (t_0, \dots, t_{i-1}, 0, t_i, \dots t_{j-1}, 0, t_j, \dots t_n)$$

Tedy $L = P$.

Nechť $\sigma \in C_{n+1}(X)$ je $(n+1)$-sing. simplex

$$\partial(\partial\sigma) = \partial\left(\sum_{i=0}^{n+1} (-1)^i \sigma \circ \varepsilon^i_{n+1}\right) = \sum_{i=0}^{n+1} (-1)^i \left(\sum_{j=0}^{n} (-1)^j (\sigma \circ \varepsilon^i_{n+1}) \circ \varepsilon^j_n\right)$$

$$= \sum_{0 \le i \le j \le n} (-1)^{i+j} \sigma \circ \varepsilon^i_{n+1} \circ \varepsilon^j_n + \sum_{0 \le j < i \le n+1} (-1)^{i+j} \sigma \circ \varepsilon^i_{n+1} \circ \varepsilon^j_n$$

1(ii)

$$= \sum_{0 \le i \le j \le n} (-1)^{i+j} \sigma \circ \varepsilon_{n+1}^{i} \circ \varepsilon_n^{j} + \sum_{0 \le l \le k \le n} (-1)^{k+1+l} \sigma \circ \underline{\varepsilon_{n+1}^{k+1} \circ \varepsilon_n^{l}}$$

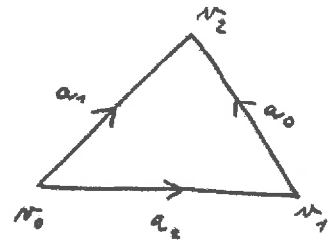
Položili jsme $i = k+1$
$j = l$

$$= \sum_{0 \le i \le j \le n} (-1)^{i+j} \sigma \circ \varepsilon_{n+1}^{i} \circ \varepsilon_n^{j} + \sum_{0 \le l \le k \le n} (-1)^{k+l+1} \sigma \circ \underline{\varepsilon_{n+1}^{l} \circ \varepsilon_n^{k}}$$

použili jsme pro $l \le k$
$\varepsilon_{n+1}^{k+1} \circ \varepsilon_n^{l} = \varepsilon_{n+1}^{l} \circ \varepsilon_n^{k}$

Sumy se liší znaménkem, proto je součet 0.

Exercise 2. *Simplicial homology of $\partial\Delta^2$.*



Simpliciální komplex

množina vrcholů V

množina simplexů S ... podmnožiny vrcholů

je-li $t \subseteq s \in S$, pak $t \in S$.

Lze geometricky realizovat v $\mathbb{R}^{n-1}$

je-li počet vrcholů n.

Simpliciální komplex X zadává řetězcový komplex

$C_i(X)$ = abelovská grupa generovaná i-simplexy

Přitom $[v_{\sigma(0)} v_{\sigma(1)} \dots v_{\sigma(i)}] = \operatorname{sign} \sigma \, [v_0 v_1 \dots v_i]$

Hraniční operátor

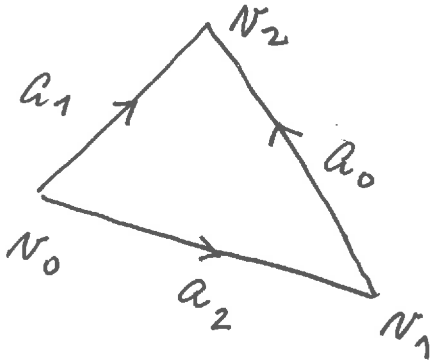
$$\partial [v_0 v_1, \dots v_i] = \sum (-1)^k [v_0, \dots, \hat{v}_k, \dots v_i]$$

Homologické grupy tohoto řetězcového komplexu se nazývají simpliciální homologické grupy.

$$H_n(X) = \frac{\ker \partial_n}{\operatorname{im} \partial_{n+1}}$$

2 (ii)

Spočítáme tyto grupy pro hranici Δ.



$a_0 = [v_1, v_2]$

$a_1 = [v_0, v_2]$

$a_2 = [v_0, v_1]$

$C_0(X) = \mathbb{Z}[v_0, v_1, v_2] \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$

$C_1(X) = \mathbb{Z}[a_0, a_1, a_2] \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$

$C_i(X) = 0$ pro $i \geq 2$

$\partial v_i = 0 \qquad \partial a_0 = v_2 - v_1, \ \partial a_1 = v_2 - v_0, \ \partial a_2 = v_1 - v_2$

$\partial \quad C_1(X) \longrightarrow C_0(X)$

Chceme spočítat $\ker \partial$ a $\operatorname{Im} \partial$

$$\begin{array}{c} \\ a_0 \\ a_1 \\ a_2 \\ \\ \end{array} \begin{array}{c} \text{hrany} \xrightarrow{\partial} \text{vrcholy} \\ \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right) \\ \qquad\qquad\qquad v_0 \quad v_1 \quad v_2 \end{array} \sim \dots \sim \begin{array}{c} \text{druhou matici} \\ \left(\begin{array}{ccc|ccc} 0 & 1 & 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 1 \\ 1 & -1 & 1 & 0 & 0 & 0 \end{array}\right) \\ \text{upravíme na schod. tvar} \end{array}$$

Tedy $\ker \partial_1 = \mathbb{Z}[a_0 - a_1 + a_2] \cong \mathbb{Z}$

$\operatorname{Im} \partial_1 = \mathbb{Z}[v_2 - v_0, v_2 - v_1] \cong \mathbb{Z} \oplus \mathbb{Z}$

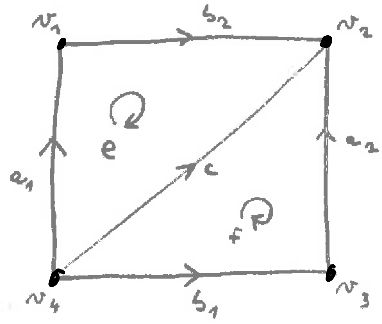
$$H_1(X) = \frac{\ker \partial_1}{\operatorname{im} \partial_2} = \ker \partial_1 \cong \mathbb{Z}$$

$$H_0(X) = \frac{\ker \partial_0}{\operatorname{im} \partial_1} = \frac{\mathbb{Z}[v_0, v_1, v_2]}{\mathbb{Z}[-v_0 + v_2, -v_1 + v_2]} = \frac{\mathbb{Z}[-v_0 + v_1, -v_0 + v_2, v_0]}{\mathbb{Z}[-v_0 + v_1, -v_0 + v_2]}$$

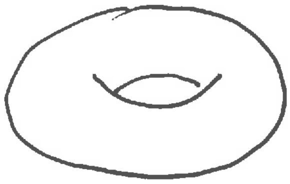
$$\cong \mathbb{Z}[v_0]$$

Exercise 3. *Simplicial complex, model of torus, compute differentials and homology.*

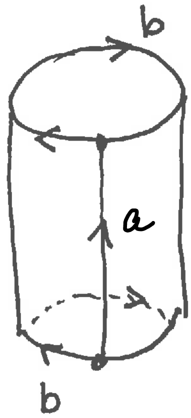
Je to model toru:



Torus



dostaneme slepením



Horní kružnici ztotožníme s dolní ve směru šipek

Zobecnění simpliciálního komplexu.

Spočítáme simpliciální homologie:

$C_0(T) = \mathbb{Z}[v]$, $\partial_0 v = 0$

$C_1(T) = \mathbb{Z}[a, b, c] \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$, $\partial_1 a = 0 = \partial_1 b = \partial_1 c$

$C_2(T) = \mathbb{Z}[e, f] \cong \mathbb{Z} \oplus \mathbb{Z}$ $\quad \partial_2 e = a + b - c$

$\partial_2 f = c - a - b$

$\ker \partial_2 = \mathbb{Z}[e+f]$

$\mathrm{im}\, \partial_2 = \mathbb{Z}[a+b-c]$

$\ker \partial_1 = \mathbb{Z}[a, b, c]$

$\mathrm{im}\, \partial_1 = 0$

$\ker \partial_0 = \mathbb{Z}[v]$

$H_2(T) = \dfrac{\ker \partial_2}{\mathrm{im}\, \partial_3} \cong \mathbb{Z}$

$H_1(T) = \dfrac{\mathbb{Z}[a, b, c]}{\mathbb{Z}[a+b-c]} \cong$

$\cong \mathbb{Z}[a, b]$

$H_0(T) \cong \mathbb{Z}$

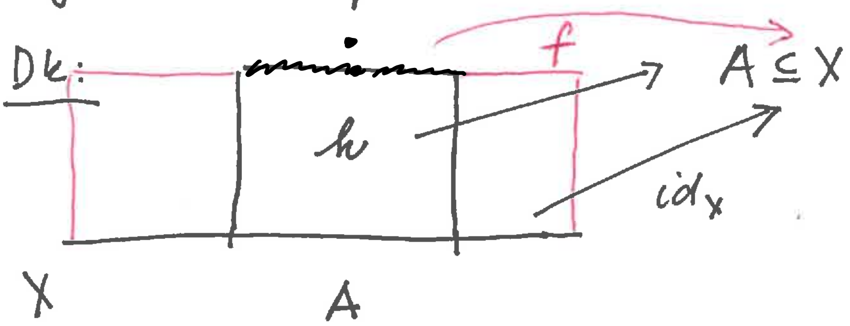
Exercise 4. *Prove the first criterion of homotopy equivalence.*

(X,A) má HEP a A je kontraktibilní v sobě, tj existuje $h : A\times[0,1] \to A$ homotopie mezi id_A a konstantním zobrazením. Pak projekce

$$q : X \longrightarrow X/A$$

je homotopická ekvivalence.

Dk:



h a id_X lze rozšířit na $f : X\times I \to X$

Definujeme $g : X/A \longrightarrow X$

$$g([x]) = f(x,1)$$

Potom $id_X \sim g\circ q$ prostřednictvím homotopie f

$$\begin{tikzcd} X\times I \arrow[r, "f"] \arrow[d, "q\times id_I"'] & X \arrow[d, "q"] \\ X/A\times I \arrow[r, dashed, "\tilde{f}"] & X/A \arrow[u, bend right, "g"'] \end{tikzcd}$$

x

$[x]$

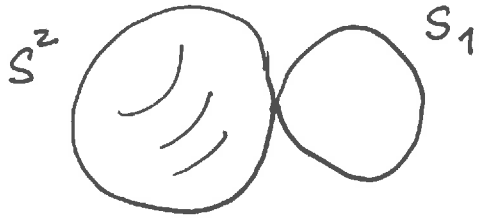
$f(A,t) \subseteq A$ proto existuje $\tilde{f} : X/A\times I \to X/A$

$\tilde{f}$ je homotopie mezi $id_{X/A}$ a $q\circ g$.

$\tilde{f}(-,0) = id_{X/A}$ $\tilde{f}(-,1) = q\circ g$

Exercise 5. $S^2 \vee S^1 \simeq S^2/S^0$ *(using First criterion)*

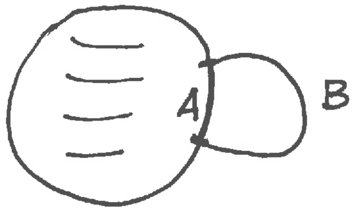
$S^2 \vee S^1$



S^2/S^0

slepíme protilehlé body na sféře

Uvažujme $X =$



sféra s přilepenou úsečkou B a úsečkou A na sféře

A i B jsou stažitelné v sobě

(X, A) i (X, B) je dvojice CW-komplexů

Podle předchozího kritéria je

$$X \simeq X/A \cong S^2 \vee S^1$$

$$X \simeq X/B \cong S^2/S^0$$

Proto i $S^2 \vee S^1$ a S^2/S^0 jsou homotopicky ekvivalentní.

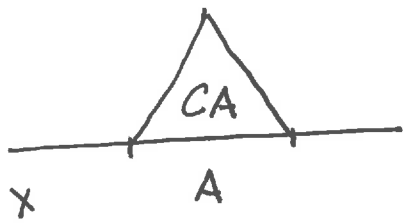
Exercise 6. *Let $i\colon A \hookrightarrow X$ is a cofibration, show $X/A \simeq X \cup CA = Ci$. (using First criterion)*

Je-li (X, A) kofibrace, je rovněž

$$(X \cup CA,\ CA)$$

kofibrace.

CA je kontraktibilní v sobě.



Na dvojici $(X\cup CA, CA)$ použijeme předchozí kriterium homotopické ekvivalence.

Dostaneme, že

$$X \cup CA \simeq X\cup CA/_{CA}$$

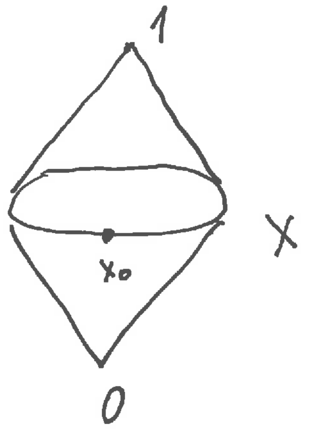
Zobrazení

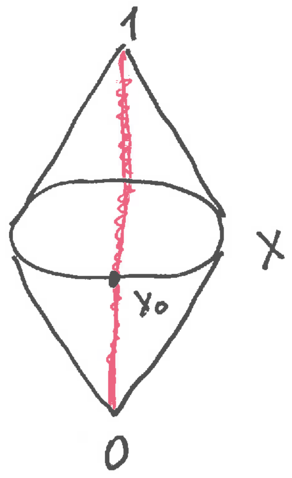
$$f : X/_A \longrightarrow X\cup CA/_{CA}$$

indukované identitou na X je homeomorfismus. Je spojité, je prosté a zobrazuje otevřené množiny na otevřené.

Exercise 7. *Application of the criterion: two types of suspensions, unreduced and reduced.*
Unreduced suspension: $SX = X \times I/\sim$, *where* $(x_1, 0) \sim (x_2, 0)$, $(x_1, 1) \sim (x_2, 2)$.
Reduced suspension: $\Sigma X = SX/\{x_0\} \times I = (X, x_0) \wedge (S^1, s_0)$
The criterion says, that if $\{x_0\} \hookrightarrow X$ *is a cofibration, then* $SX \simeq \Sigma X$.

$$I \simeq \{(x_0, t), t \in I\} \subseteq SX \longrightarrow SX/\{x_0, t), t \in I\} = \Sigma X$$

$SX =$ 



$\Sigma X = SX/_{[0,1]\times x_0}$

$$SX = [0,1]\times X/\sim \qquad \begin{matrix}(0, x) \sim (0, y)\\ (1, x) \sim (1, y)\end{matrix}$$

Exercise 8. *There is a lemma, that says: Given the following diagram, where rows are long exact sequences and m is iso,*

$$\begin{tikzcd}
K_n \arrow[r, "f"] \arrow[d, "k"] & L_n \arrow[r, "g"] \arrow[d, "l"] & M_n \arrow[r, "h"] \arrow[d, "m"] & K_{n-1} \arrow[r] \arrow[d] & L_{n-1} \arrow[r] \arrow[d] & M_{n-1} \arrow[d] \\
\overline{K}_n \arrow[r, "\bar f"] & \overline{L}_n \arrow[r, "\bar g"] & \overline{M}_n \arrow[r, "\bar h"] & \overline{K}_{n-1} \arrow[r] & \overline{L}_{n-1} \arrow[r] & \overline{M}_{n-1}
\end{tikzcd}$$

we get a long exact sequence

$$K_n \xrightarrow{f\oplus k} L_n \oplus \overline{K}_n \xrightarrow{l-\bar f} \overline{L}_n \xrightarrow{\partial_*} K_{n-1} \longrightarrow \cdots$$

where $\partial_* = h \circ m^{-1} \circ \bar g$. *Show exactness in* $L_n \oplus \overline{K}_n$ *and in* $\overline{L}_n$.

$\boxed{\text{Exaktnost v } L_n \oplus \overline{K}_n}$

Platí $(l-\bar f)\circ(f\oplus k) = l\circ f - \bar f\circ k = 0$

Proto $\ker(l-\bar f) \supseteq \operatorname{im} f\oplus k$.

$$\begin{tikzcd}
c \arrow[r, mapsto] & f(c) & a \arrow[r, mapsto] & b \arrow[r, mapsto] & 0 \\
M_{n+1} \arrow[r, "f"] \arrow[d, "m"] & K_n \arrow[r, "g"] \arrow[d, "k"] & L_n \arrow[r, "h"] \arrow[d, "l"] & M_n \arrow[d, "m"] \\
\overline{M}_{n+1} \arrow[r, "\bar f"] & \overline{K}_n \arrow[r, "\bar g"] & \overline{L}_n \arrow[r, "\bar h"] & \overline{M}_n \\
\bar c \arrow[r, mapsto] & \bar a - k(a) & \bar a \arrow[r, mapsto] & \bar b \arrow[r, mapsto] & 0
\end{tikzcd}$$

$a - f(c)$

$(b, \bar a) \in L_n \oplus \overline{K}_n$ $\quad (b, \bar a) \in \ker(l - \bar f)$

Pak $\bar h(\bar b) = 0$ a z exaktnosti a $h(b) = 0$ neboť m je izo. Existuje $a \in K_n$, $g(a) = b$

Uvažujme $\bar a - k(a)$. Pak $\bar g(\bar a - k(a)) = \bar g(\bar a) - \bar g k(a) =$

$= \bar b - l g(a) = \bar b - l(b) = \bar b - \bar b = 0$.

Existuje proto $\bar c \in \overline{M}_{n+1}$ a $c = m^{-1}(\bar c) \in M_n$.

Potom $a - f(c) \in K_n$ se zobrazí

$$g(a) - g(f(c)) = b - 0 = b$$

$$k(a - f(c)) = k(a) - k f(c) = k(a) - m \bar f(\bar c)$$

$$= k(a) - \bar a - k(a) = \bar a$$

Tedy $(b, \bar a) \in \operatorname{im}(f \oplus k)$.

Exaktnost v $\bar{L}_n$

Platí $(h \circ m^{-1} \circ \bar{g}) \circ (l - \bar{f}) = h \circ m^{-1} \circ \bar{g} \circ l - h \circ m^{-1} \circ \underbrace{\bar{g}\,\bar{f}}_{0}$

$$= \underbrace{\bar{h} \circ \bar{g}}_{0} \circ l = 0$$

Tedy $\quad \operatorname{Im}(l - \bar{f}) \subseteq \ker \partial_*$

Obrácená inkluze.

Nechť $\quad \bar{b} \in \ker \partial_* \subseteq \bar{L}_n$

$$\begin{tikzcd}
 & b \arrow[r, mapsto] & c \arrow[r, mapsto] \arrow[dd, bend left, mapsto] & 0 \\
K_n \arrow[r] \arrow[d] & L_n \arrow[r, "g"] \arrow[d, "l"] & M_n \arrow[r] \arrow[d] & K_{n-1} \arrow[d] \\
\bar{K}_n \arrow[r, "\bar{f}"] & \bar{L}_n \arrow[r, "\bar{g}"] & \bar{M}_n \arrow[r, "\bar{h}"] & \bar{K}_{n-1} \\
 & \bar{b} \arrow[r, mapsto] & \bar{c} \arrow[r, mapsto] & 0 \\
\bar{a} \arrow[r, mapsto] & l(b) - \bar{b} \arrow[r, mapsto] & 0 &
\end{tikzcd}$$

$\bar{g}(\bar{b}) = \bar{c}$, $\quad m^{-1}(\bar{c}) = c$, $\quad h(c) = 0$ neboť $\partial_*(\bar{b}) = 0$.

Z exaktnosti, $\bar{h}(\bar{c}) = 0$.

Existuje $\quad b \in L_n$, že $\quad g(b) = c$.

$l(b) - \bar{b}$ je v jádru $\bar{g}$: $\quad \bar{g}(l(b)) - \bar{g}(\bar{b}) = m\, g(b) - \bar{c}$

$$= \bar{c} - \bar{c} = 0.$$

Existuje $\quad \bar{a} \in \bar{K}_n$, že $\quad \bar{f}(\bar{a}) = l(b) - \bar{b}$

Potom $\quad (l - \bar{f})(b, \bar{a}) = l(b) - \bar{f}(\bar{a}) = l(b) - (l(b) - \bar{b}) = \bar{b}$

$\bar{b} \in \operatorname{im}(l - \bar{f})$