Exercise 1. *Show $\partial\partial = 0$. Use formula $\varepsilon^i_{n+1} \circ \varepsilon^j_n = \varepsilon^{j+1}_{n+1} \circ \varepsilon^i_n$, where $i < j$. The definition for $\sigma \in C_n(X)$, $\sigma \colon \Delta^n \to X$, is*

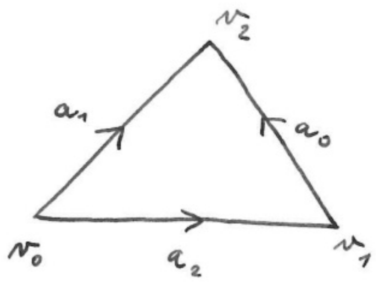
$$\partial\sigma = \sum_{i=0}^{n} (-1)^i \, \sigma \circ \varepsilon^i_n.$$

Solution. Easily workout

$$\begin{aligned}\partial(\partial\sigma) &= \partial\Big(\sum_{i=0}^{n+1}(-1)^i \, \sigma \circ \varepsilon^i_{n+1}\Big) = \sum_{i=0}^{n+1}(-1)^i \, \partial(\sigma \circ \varepsilon^i_{n+1}) = \\ &= \sum_{i=0}^{n+1}(-1)^i \Big(\sum_{j<i}(-1)^j \sigma \circ \varepsilon^i_{n+1} \circ \varepsilon^j_n + \sum_{j>i}(-1)^{j-1}\sigma \circ \varepsilon^i_{n+1} \circ \varepsilon^{j-1}_n\Big) = \\ &= \sum_{i=1}^{n+1}\sum_{j<i}(-1)^i(-1)^j \sigma \circ \varepsilon^i_{n+1} \circ \varepsilon^j_n + \sum_{i=1}^{n}\sum_{j>i}(-1)^{j+i-1}\sigma \circ \varepsilon^i_{n+1} \circ \varepsilon^{j-1}_n,\end{aligned}$$

now, with proper reindex and shift, this yields $\varepsilon^i_{n+1} \circ \varepsilon^j_n = \varepsilon^{j+1}_{n+1} \circ \varepsilon^i_n = \varepsilon^j_{n+1} \circ \varepsilon^{i-1}_n$, both sums are of the same elements but with opposite signs. Hence, $\partial\partial = 0$. □

Exercise 2. *Simplicial homology of $\partial\Delta^2$.*



Solution. Chain complex of this simplicial homology is $C_0 = \mathbb{Z}[v_0, v_1, v_2] = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$, $C_1 = \mathbb{Z}[a_0, a_1, a_2] = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. So

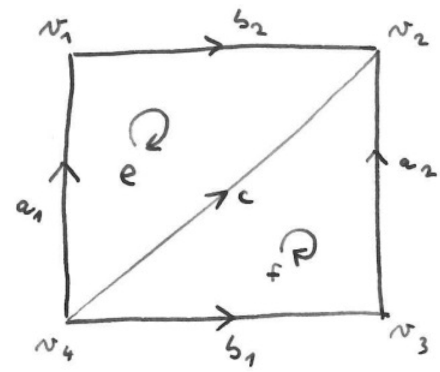
$$0 \to C_1 \xrightarrow{\partial} C_0 \to 0,$$

where we want to determine ∂ and we know $\partial a_0 = v_2 - v_1$, $\partial a_1 = v_2 - v_0$, $\partial a_2 = v_1 - v_0$. Using simple linear algebra, we study generators $\ker\partial$ and $\mathrm{im}\partial$:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 1 & -1 & 1 & 0 & 0 & 0 \end{array}\right),$$

therefore $\ker\partial$ has a generator $a_0 - a_1 + a_2$ and $\mathrm{im}\partial$ has two generators $-v_1 + v_2$ and $-v_0 + v_2$. We get $H_0 = \dfrac{\mathbb{Z}[v_0, v_1, v_2]}{\mathbb{Z}[-v_1 + v_2, -v_0 + v_2]} = \dfrac{\mathbb{Z}[-v_1 + v_2, v_0 + v_2, v_2]}{\mathbb{Z}[-v_1 + v_2, -v_0 + v_2]} = \mathbb{Z}[v_0] = \mathbb{Z}$ and $H_1 = \ker\partial = \mathbb{Z}[a_0 - a_1 + a_2] = \mathbb{Z}$. □

Exercise 3. *Simplicial complex, model of torus, compute differentials and homology.*



Solution. Again, we get simplicial chain complex C_* formed by free abelian groups generated by equivalence classes of simplicies. Note a_1, a_2 are actually one generator, same for b_1, b_2. All the vertices are also equivalent. We choose the orientation and fix it.

Thus we get $C_0 = \mathbb{Z}[v] = \mathbb{Z}, C_1 = \mathbb{Z}[a,b,c] = \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}, C_2 = \mathbb{Z}[e,f] = \mathbb{Z}\oplus\mathbb{Z}, C_3 = 0$, and the following holds: $\partial a = 0, \partial b = 0, \partial c = 0$, as well as $\partial e = a+b-c, \partial f = c-a-b, \partial(e+f) = 0$, so we get $\ker\partial = \mathbb{Z}[e+f], \mathrm{im}\partial = \mathbb{Z}[a+b-c]$.

Let T be the torus. Then

$$\begin{aligned} H_2(T) &= \ker\partial_2 = \mathbb{Z}[e+f] = \mathbb{Z}, \\ H_1(T) &= \mathbb{Z}[a,b,c]/\mathbb{Z}[a+b-c] = \frac{\mathbb{Z}[a,b,a+b-c]}{\mathbb{Z}[a+b-c]} = \mathbb{Z}[a,b] = \mathbb{Z}\oplus\mathbb{Z}, \\ H_0(T) &= \ker\partial_0 = \mathbb{Z}. \end{aligned}$$

□

Exercise 4. *Prove the first criterion of homotopy equivalence.*

Solution. We take $h\colon A\times I \to A$, on $A\times\{0\}$ it is identity on A and constant on $A\times\{1\}$.

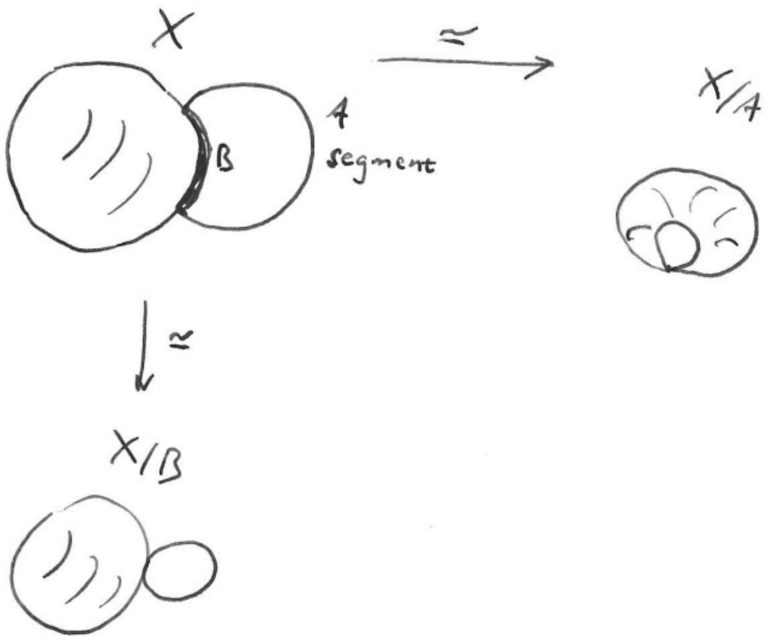
$$\begin{array}{ccc} X\times I & \xrightarrow{f} & X \\ {\scriptstyle q\times\mathrm{id}_I}\downarrow & & \downarrow{\scriptstyle q} \\ X/A\times I & \xrightarrow{\bar{f}} & X/A \end{array}$$

and find $g\colon X/A\to X$. Define $\bar{f}(x,t) = f(x,t)$, $\bar{f}([x],t) = [f(x,t)]$. If we define $g\colon X/A \to A, [x]\mapsto f(x,1)$, then it is well defined. Now we want to show, that the compositions are homotopy equivalent to the identities.

$g\circ q\sim \mathrm{id}_X$: $g(q(x)) = g([x]) = f(x,1)$, just the way we defined it, so f is the homotopy, as $f(-,0) = \mathrm{id}_X$ and $f(-,1) = g\circ q$,

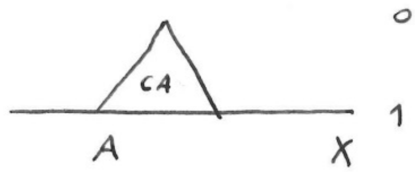
$q\circ g\sim\mathrm{id}_{X/A}$: $q(g([x])) = q(f(x,1)) = [f(x,1)] = \bar{f}([x],1)$ and $\mathrm{id}_{X/A} = \bar{f}([x],0)$, so in this case the map $\bar{f}$ is homotopy. □

Exercise 5. $S^2 \vee S^1 \simeq S^2/S^0$ *(using First criterion)*



Solution. In the picture (hopefully) above, A is a segment as well as B, so contractible in itself. Clearly $S^2 \vee S^1 = X/B$ and $S^2/S^0 = X/A$ and $X \simeq X/A$ and $X \simeq X/B$ by criterion, therefore $X/A \simeq X/B$ and we are done. □

Exercise 6. *Let* $i\colon A \hookrightarrow X$ *is a cofibration, show* $X/A \simeq X \cup CA = Ci$. *(using First criterion)*



Solution. We know $CA \hookrightarrow X \cup CA$ is a cofibration using homework 1, exercise 2, with $Y = CA$. Then by criterion $X \cup CA \simeq X \cup CA/CA$. Also X/A is homeomorphic to $X \cup CA/CA$ (see picture above), which concludes the result.

□

Exercise 7. *Application of the criterion: two types of suspensions, unreduced and reduced.*

Unreduced suspension: $SX = X \times I/\sim$, *where* $(x_1, 0) \sim (x_2, 0)$, $(x_1, 1) \sim (x_2, 2)$.
Reduced suspension: $\Sigma X = SX/\{x_0\} \times I = (X, x_0) \wedge (S^1, s_0)$
(this might be a homework)
The criterion says, that if $\{x_0\} \hookrightarrow X$ *is a cofibration, then* $SX \simeq \Sigma X$.

$$I \simeq \{(x_0, t), t \in I\} \subseteq SX \longrightarrow SX/\{x_0, t), t \in I\} = \Sigma X$$

Exercise 8. *Given the following diagram, where rows are long exact sequences and* m *is an iso*

$$\begin{array}{ccccccccccc} K_n & \xrightarrow{i} & L_n & \xrightarrow{j} & M_n & \xrightarrow{h} & K_{n-1} & \longrightarrow & L_{n-1} & \longrightarrow & M_{n-1} \\ \downarrow f & & \downarrow g & & \downarrow m & & \downarrow & & \downarrow & & \downarrow \\ \overline{K}_n & \xrightarrow{\bar{i}} & \overline{L}_n & \xrightarrow{\bar{j}} & \overline{M}_n & \longrightarrow & \overline{K}_{n-1} & \longrightarrow & \overline{L}_{n-1} & \longrightarrow & \overline{M}_{n-1} \end{array}$$

we get a long exact sequence

$$K_n \xrightarrow{(i,f)} L_n \oplus \overline{K}_n \xrightarrow{g-\bar{i}} \overline{L}_n \xrightarrow{\partial_*} K_{n-1} \longrightarrow \cdots$$

We can denote $\partial_* = h \circ m^{-1} \circ \bar{j}$.
Show exactness in $L_n \oplus \bar{K}_n$ *and also in* $\bar{L}_n$.

Solution. We have $(g - \bar{i}) \circ (i, f) = \bar{i}f - gi = 0$ obviously. For $x \in L_n, y \in \bar{K}_n$ we have $(g - \bar{i})(x, y) = 0$, so $g(x) = \bar{i}(y)$. Now, let x be such that $j(x) = 0$, then there is $z \in K_n$ such that $i(z) = x$. Then, suppose $g(x) = a \in \bar{L}_n$, then by m being iso we know $\bar{j}(a) = 0$, so exists $y \in \bar{K}_n$ such that $\bar{i}(y) = a$. Since $f(z)$ and y have the same image, their difference has a preimage, i.e. exists $b \in \bar{M}_{n+1}$ such that $b \mapsto y - f(z)$. By iso then there exists $c \mapsto z$, or denote $h(c) = z$. Now, all of this is much easier with a picture (that I don't draw). Compute now:
$f(z + c) = f(z) + y - f(z) = y$ and $i(z + h(c)) = i(z) = x$, and we are done.

Exactness in $\bar{L}_n$ is easier. It holds $\partial \circ (g - \bar{i}) = 0$, so take $x \in \ker \partial$ (also, $x \in \bar{L}_n$). Now, $x \mapsto a$, by iso there is b in the upper row that maps to zero. Then there exists y such that $y \mapsto b$. Now we can work with $x - g(y)$. There exists also z such that, obviously, $z \mapsto x - g(y) \mapsto a - a = 0$. Get $x = g(y) + \bar{i}(z) = g(y) - \bar{i}(-z)$, that is we needed to express x as this difference, hence we are done. □