Exercise 1. *Prove 5-lemma.*

Mějme komutativní diagram abelovských grup

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, "i", "\cong"'] & B \arrow[r, "\beta"] \arrow[d, "j", "\cong"'] & C \arrow[r, "\varkappa"] \arrow[d, "k"] & D \arrow[r, "\delta"] \arrow[d, "l", "\cong"'] & E \arrow[d, "m", "\cong"'] \\
\bar{A} \arrow[r, "\bar\alpha"'] & \bar{B} \arrow[r, "\bar\beta"'] & \bar{C} \arrow[r, "\bar\varkappa"'] & \bar{D} \arrow[r, "\bar\delta"'] & \bar{E}
\end{tikzcd}$$

Jestliže jsou horizontální posloupnosti exaktní a 1., 2., 4. a 5. vertikální homo je izo, pak i třetí vertikální homo je izo.

① k je mono

$$\begin{tikzcd}
a \arrow[r, mapsto] & b \arrow[r, mapsto] & c \arrow[r, mapsto] & 0 \\
A \arrow[r, "\alpha"] \arrow[d, "i", "\cong"'] & B \arrow[r, "\beta"] \arrow[d, "j", "\cong"'] & C \arrow[r, "\varkappa"] \arrow[d, "k"] & D \arrow[d, "\cong"] \\
\bar{A} \arrow[r, "\bar\alpha"'] & \bar{B} \arrow[r] & \bar{C} \arrow[r] & \bar{D} \\
\bar{a} \arrow[r, mapsto] & \bar{b} \arrow[r, mapsto] & 0 \arrow[r, mapsto] & 0
\end{tikzcd}$$

- $c \in C$, $k(c) = 0$
  Pak $\varkappa(c) = 0$
- $\exists\, b \in B$, že $\beta(b) = c$
  $j(b) = \bar{b}$, $\bar\beta(\bar{b}) = 0$
- $\exists\, \bar{a} \in \bar{A}$ $\bar\alpha(\bar{a}) = \bar{b}$
  $\exists\, a \in A$ $i(a) = \bar{a}$
  $\alpha(a) = b$ neboť j je izo
- $0 = \beta(\alpha(a)) = \beta(b) = c$

což jsme chtěli dokázat.

② k je epi

$$\begin{tikzcd}
b & c+\beta(b) \quad c \arrow[r, mapsto] & d & 0 \\
B \arrow[r, "\beta"] \arrow[d, "j", "\cong"'] & C \arrow[r, "\varkappa"] \arrow[d, "k"] & D \arrow[r, "\delta"] \arrow[d, "l", "\cong"'] & E \arrow[d, "m", "\cong"'] \\
\bar{B} \arrow[r, "\bar\beta"'] & \bar{C} \arrow[r, "\bar\varkappa"'] & \bar{D} \arrow[r, "\bar\delta"'] & \bar{E} \\
\bar{b} & \bar{c} \arrow[r, mapsto] & \bar{d} \arrow[r, mapsto] & 0 \\
\bar{b} \arrow[r, mapsto] & \bar{c}-k(c) \arrow[r, mapsto] & 0 &
\end{tikzcd}$$

- $\bar{c} \in \bar{C}$
  $\bar\varkappa(\bar{c}) = \bar{d}$, $\bar\delta\bar\varkappa(c) = 0$
- $\exists\, d \in D$ $\delta(d) = 0$
- $\exists\, c \in C$ $\varkappa(c) = d$
- $\bar{c} - k(c) \in \bar{C}$ $\quad c - k(c) \in \ker \bar\varkappa$

1 (ii)

- $\exists\, \bar{b} \in \bar{B} \quad \bar{\beta} = \bar{c} - k(c)$
- $\exists\, b \in B$
- $c + \beta(b) \in C$ se zobrazí na $\bar{c}$

$$k(c + \beta(b)) = k(c) + k\beta(b) = k(c) + \bar{\beta} j(b) =$$
$$= k(c) + \bar{\beta}(\bar{b}) = k(c) + \bar{c} - k(c) = \bar{c}$$

Exercise 2. *There is a long exact sequence of the triple (X, A, B), i.e. $(B \subseteq A \subseteq X)$:*

$$\cdots \to H_n(A,B) \xrightarrow{i} H_n(X,B) \xrightarrow{j_X} H_n(X,A) \xrightarrow{D_*} H_{n-1}(A,B) \to \cdots,$$

with $H_n(X,A) \xrightarrow{\partial_*} H_{n-1}(A) \xrightarrow{j_A} H_{n-1}(A,B)$. We get this sequence from a special short exact sequence of chain complexes. Show that it is exact and that the triangle commutes, that is $D_* = j_A \circ \partial_*$.

Krátká exaktní posloupnost

$$0 \to \frac{C_*(A)}{C_*(B)} \xrightarrow{i} \frac{C_*(X)}{C_*(B)} \xrightarrow{j} \frac{C_*(X)}{C_*(A)} \to 0$$

i je prosté

J je na

$j \circ i = 0$

$\ker j \subseteq \operatorname{im} i$

Krátká ex. posloupnost indukuje dlouhou.

Musíme ukázat, že $D_* = j_A \circ \partial_*$ je svazující homomorfismus v této posloupnosti.

Vezmeme $c \in C_*(X)$ s hranicí v $C_*(A)$. $[c] \in \frac{C_*(X)}{C_*(A)}$

Vezmeme stejné c, to reprezentuje prvek $\frac{C_*(X)}{C_*(B)}$

$$\begin{tikzcd}
 & c & c \\
 & \frac{C_n(X)}{C_n(B)} \arrow[r] \arrow[d] & \frac{C_n(X)}{C_n(A)} \\
C_{n-1}(A) \arrow[d, "j_A"] & & \\
\frac{C_{n-1}(A)}{C_{n-1}(B)} \arrow[r] & \frac{C_{n-1}(X)}{C_{n-1}(B)} & \partial c \in C_{n-1}(A) \\
[\partial c]_{A,B} & \partial c &
\end{tikzcd}$$

$$D_* = j_A \circ \partial_*$$

Exercise 3. *Apply previous exercise to the triple $(D^k, S^{k-1}, *)$, where $*$ is a point.*

$$\begin{tikzcd}
H_n(S^{k-1},*) \arrow[r] & H_n(D^k,*) \arrow[r] \arrow[d, phantom, "\shortparallel"] & H_n(D^k,S^{k-1}) \arrow[r, "\varphi"] & H_{n-1}(S^{k-1},*) \arrow[d] \\
 & 0 & 0 = & H_{n-1}(D^k,*)
\end{tikzcd}$$

Odtud plyne, že φ (svazující homo) je izomorfismus

$$H_n(D^k,S^{k-1}) \cong H_{n-1}(S^{k-1},*) = \overline{H}_{n-1}(S^k)$$

Z definice lze spočítat, že

$$H_0(S^0,*) \cong \mathbb{Z}, \quad H_n(S^0,*) \cong 0, \ n \neq 0$$

Všimněte si, že dvojice (D^k, S^{k-1}) a $(\Delta^k, \partial\Delta^k)$ jsou homeomorfní. Odtud a s využitím další úlohy dostaneme

$$H_n(D^k,S^{k-1}) \cong \overline{H}_{n-1}(S^{k-1}) \cong \begin{cases} \mathbb{Z} & n=k \\ 0 & \text{jinak} \end{cases}$$

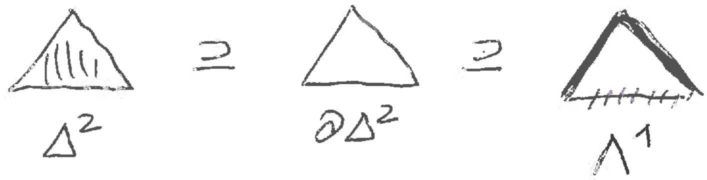
Exercise 4. *Show that the chain in $C_k(\Delta^k, \partial\Delta^k)$ given by $\mathrm{id}\colon \Delta^k \to \Delta^k$ is the representative of the generator of*

$$H_k(\Delta^k, \partial\Delta^k) \cong \mathbb{Z}.$$

(Use induction and the long exact sequence for triple.)

$$\mathrm{id} \in \frac{C_k(\Delta^k)}{C_k(\partial\Delta^k)}$$

Nechť Λ^{k-1} je hranice $\partial\Delta^k$ bez jedné stěny (spodní)



Z dlouhé ex. posloupnosti

$$\underset{0}{\underset{\parallel}{H_n(\Delta^k, \Lambda^{k-1})}} \longrightarrow H_n(\Delta^k, \partial\Delta^k) \xrightarrow{\cong} H_{n-1}(\partial\Delta^k, \Lambda^{k-1}) \longrightarrow \underset{0}{\underset{\parallel}{H_{n-1}(\Delta^k, \Lambda^{k-1})}}$$

Aplikujeme větu o výřezu na dvojici $\partial\Delta^k, \Lambda^{k-1}$

$$H_{n-1}(\partial\Delta^k \setminus C, \Lambda^{k-1} \setminus C) \cong H_{n-1}(\partial\Delta^k, \Lambda^{k-1})$$

$$H_{n-1}(\partial\Delta^k \setminus C, \Lambda^{k-1} \setminus C) \cong H_{n-1}(\Delta^{k-1}, \partial\Delta^{k-1})$$

Proto

$$H_n(\Delta^k, \partial\Delta^k) \xrightarrow{\cong} H_{n-1}(\Delta^{k-1}, \partial\Delta^{k-1})$$

Z předchozí úlohy víme, že

$$H_n(\Delta^1, \partial\Delta^1) \cong H_{n-1}(\partial\Delta^1, *) = \begin{matrix} \mathbb{Z} & n=1 \\ 0 & n \neq 1 \end{matrix}$$

Proto

$$H_n(\Delta^k, \partial\Delta^k) \cong \begin{matrix} \mathbb{Z} & n=k \\ 0 & \text{jinak} \end{matrix}$$

Singulární simplex $id_k\ \Delta^k \to \Delta^k$ má hranici $\partial id \in C_{k-1}(\partial\Delta^k)$. Proto je cyklem v $C_k(\Delta^k, \partial\Delta^{k-1})$. Indukcí ukážeme, že je generátor.

id_k reprezentuje třídu v $H_k(\Delta^k, \partial\Delta^k)$

$$H_k(\Delta^k, \partial\Delta^k) \longrightarrow H_{k-1}(\partial\Delta^k, \Lambda^{k-1})$$

Svazující homomorfismus na úrovni řetězců, zobrazuje id_k na $\partial id_k \in C_{k-1}(\partial\Delta^k)$

~~kde~~ Na úrovni výřezu

$$H_{k-1}(\Delta^{k-1}, \partial\Delta^{k-1}) \longrightarrow H_k\ \partial\Delta^k, \Lambda^{k-1})$$

reprezentuje id_{k-1} $\Delta^{k-1} \to \Delta^{k-1}$ stejný prvek jako $\partial id_k \in C_{k-1}(\partial\Delta^k, \Lambda^{k-1})$.

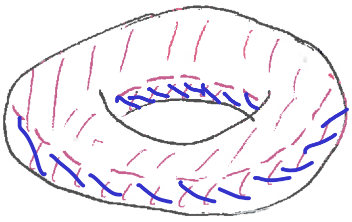
Je-li tedy $[id_{k-1}]$ generátor v $H_{k-1}(\Delta^{k-1}, \partial\Delta^{k-1})$ musí být $[id_k]$ generátor v $H_k(\Delta^k, \partial\Delta^k)$.

Zbývá dokázat, že $id_1 : [0,1] \to [0,1]$ reprezentuje generátor v $H_1(\Delta^1, \partial\Delta^1)$. Víme, že

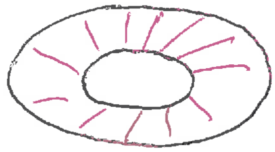
$$H_1(\Delta^1, \partial\Delta^1) \xrightarrow{\cong} H_0(\{0,1\}, \{0\})$$

je izo. $[id_1]$ se zobrazuje $[1]$, což je generátor v $H_0(\{0,1\}, \{0\})$.

Exercise 5. *Using the Mayor-Vietoris exact sequence compute the homology groups of the torus. (note: Vietoris died in 2002, aged 110, remarkable)*



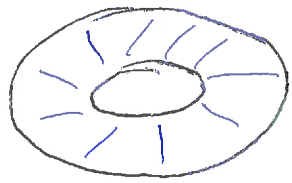
A ... horní část tory



$\simeq S^1$

B ... dolní část toru

Torus $X = A \cup B$



$\simeq S^1$

$A \cap B = S^1 \sqcup S^1$

Mayerova – Vietorisova posloupnost

$$H_n(A\cap B) \longrightarrow H_n(A)\oplus H_n(B) \longrightarrow H_n(X) \longrightarrow H_{n-1}(A\cap B)$$

$n = 2$

$$\underset{\overset{\parallel}{0}}{H_2(A)\oplus H_2(B)} \longrightarrow H_2(X) \longrightarrow \underset{\mathbb{Z}\oplus\mathbb{Z}}{H_1(A\cap B)} \xrightarrow{f} \underset{\mathbb{Z}\oplus\mathbb{Z}}{H_1(A)\oplus H_1(B)}$$

$$\longrightarrow \underset{\mathbb{Z}}{H_1(X)} \longrightarrow \underset{\mathbb{Z}\oplus\mathbb{Z}}{H_0(A\cap B)} \xrightarrow{g} \underset{\mathbb{Z}\oplus\mathbb{Z}}{H_0(A)\oplus H_0(B)}$$

$$\longrightarrow H_0(X) \longrightarrow 0$$

X je souvislý $\quad H_0(X) = \mathbb{Z}$

$H_2(X) \cong \ker f \quad , \quad f : \mathbb{Z}\oplus\mathbb{Z} \longrightarrow \mathbb{Z}\oplus\mathbb{Z}$

$\qquad (a, b) \longrightarrow (a+b, a+b)$

$\ker f = \{(a,-a), a \in \mathbb{Z}\} \cong \mathbb{Z} \qquad H_2(X) \cong \mathbb{Z}$

Dále $\quad 0 \to \operatorname{im} f \cong \mathbb{Z} \longrightarrow H_1(X) \longrightarrow \underset{\cong \mathbb{Z}}{\ker g} \longrightarrow 0$

Proto $\quad H_1(X) \cong \mathbb{Z}\oplus\mathbb{Z}$