Exercise 1. *Prove 5-lemma.*

Exercise 2. *There is a long exact sequence of the triple (X, A, B), i.e. $(B \subseteq A \subseteq X)$:*

$$\cdots \to H_n(A,B) \xrightarrow{i} H_n(X,B) \xrightarrow{j_X} H_n(X,A) \xrightarrow{D_*} H_{n-1}(A,B) \to \cdots,$$

with $H_n(X,A) \xrightarrow{\partial_*} H_{n-1}(A) \xrightarrow{j_A} H_{n-1}(A,B)$. We get this sequence from a special short exact sequence of chain complexes. Show that it is exact and that the triangle commutes, that is $D_* = j_A \circ \partial_*$.

Solution. The chain complex of a pair is a quotient. We have

$$0 \longrightarrow \frac{C_*(A)}{C_*(B)} \xrightarrow{i} \frac{C_*(X)}{C_*(B)} \xrightarrow{j} \frac{C_*(X)}{C_*(A)} \to 0.$$

Take $c \in C_*(A)$, then $ji[c] = j[ic] = [jic] = [c]$, but seen as a different class. So, $j \circ i = 0$ and inclusion $\operatorname{im} i \subseteq \ker j$ holds. The other inclusion $\ker j \subseteq \operatorname{im} i$ is obvious.

Analogous: $0 \to C_*(A) \to C_*(X) \to \dfrac{C_*(X)}{C_*(A)} \to 0.$

Now, show D_* equals the composition:
Note, if you are familiar with the definition, it's clear.
Take a chain complex in $C_*(X)$ with boundary 0, chain in $C_*(A)$, take preimage, boundary in $C_*(B)$, we have $[c] \in C_n(X,B)$ and its image $[c] \in C_n(X,A)$, same representative, but different equivalence. Take $\partial c \in C_{n-1}(X,B)$, it has preimage $[\partial c]_{A,B} \in C_{n-1}(A,B)$. (Drawing diagram and chase helps.) The equality of D_*, that it is composition, holds, basically thanks to j_A being inclusion. □

Exercise 3. *Apply previous exercise to the triple $(D^k, S^{k-1}, *)$, where $*$ is a point.*

Solution.

$$\cdots H_n(S^{k-1},*) \to H_n(D^k,*) \to H_n(D^k,S^{k-1}) \xrightarrow{\varphi} H_{n-1}(S^{k-1},*) \to H_{n-1}(D^k,*) \to \cdots,$$

and, note $H_n(*) = \mathbb{Z}$ for $n = 0$ and $H_n(*) = 0$ otherwise.

We also work with reduced homology groups: $\bar{H}_n(X) = H_n(X, x_0)$, $H_*(X) = \bar{H}_*(X) \oplus H_*(*)$. Since D^k is contractible, $H_n(D^k,*) = \bar{H}_n(D^k) = 0$ and $H_{n-1}(D^k,*) = 0$, so we have

$$\cdots H_n(S^{k-1},*) \to 0 \to H_n(D^k,S^{k-1}) \xrightarrow{\varphi} H_{n-1}(S^{k-1},*) \to 0 \to \cdots,$$

where φ is *iso*.

Reduced homology for pairs is the same as unreduced:

$$H_n(D^k,S^{k-1}) = \bar{H}_n(D^k,S^{k-1}) \cong \bar{H}_{n-1}(S^{k-1}).$$

With this we know that $H_n(D^k,S^{k-1}) = \bar{H}_n(D^k,S^{k-1}) = \mathbb{Z}$ for $n = k$ and it is 0 for $n \neq k$.

Note also, that $(D^k,S^{k-1}) \cong (\Delta^k, \partial\Delta^k)$, this might be useful later on. □

Exercise 4. *Show that the chain in* $C_k(\Delta^k, \partial\Delta^k)$ *given by* $\mathrm{id}\colon \Delta^k \to \Delta^k$ *is the representative of the generator of*

$$H_k(\Delta^k, \partial\Delta^k) \cong \mathbb{Z}.$$

(Use induction and the long exact sequence for triple.)

Solution. First denote $\wedge^{k-1}$ boundary without interior of one face. Then work with the triple $\Delta^k, \partial\Delta^k, \wedge^{k-1}$, so the sequence needed is as follows:

$$0 \to H_k(\Delta^k, \partial\Delta^k) = \mathbb{Z} \to H_{k-1}(\partial\Delta^k, \wedge^{k-1}) \to 0,$$

where we have the zeroes because $\Delta^k, \wedge^{k-1}$ are contractible to points. Use excision theorem, $H_*(X - C, A - C) \cong H_*(X, A)$, where C is the boundary with bottom cut out (imagine upper part of the letter Δ, i.e. triangle). We know that $H_{k-1}(\Delta^{k-1}, \partial\Delta^{k-1}) = \mathbb{Z}$ and this is isomorphic (by excision theorem) to $H_{k-1}(\partial\Delta^k, \wedge^{k-1})$. So, everything is $\mathbb{Z}$. We want to show that images of id_{k-1} and id_k are the same, this is actually the inductive step.

Suppose that the generator in $H_{k-1}(\Delta^{k-1}, \partial\Delta^{k-1})$ is given by $\mathrm{id}_{k-1}\colon \Delta^{k-1} \to \Delta^{k-1}$. Then $\mathrm{id}_k\colon \Delta^k \to \Delta^k$ is cycle again, represents element $[\mathrm{id}_k] \in H_k(\Delta^k, \partial\Delta^k)$.
The Beginning of the Induction *(coming to theaters this summer)*:

$$H_1([-1,1], \{-1,1\}) \xrightarrow{\partial_*} H_0(\{-1,1\}, \{1\}),$$

take $\mathrm{id}\colon [-1,1] \to [-1,1]$ as a chain complex, $1 - (-1) = [-1]$, $[-1]$ generator. □

Exercise 5. *Using the Mayor-Vietoris exact sequence compute the homology groups of the torus. (note: Vietoris died in 2002, aged 110, remarkable)*

Solution. It goes as union, intersection, pair and we will want to determine the union. First draw two disks with holes (these glued together give a torus). Call one interior of the disk A and the other B. We work with $X = A \cup B$, it is not a problem, that we need to work with A, B open, as from the point of view of homology it doesn't matter.

The sequence is

$$\cdots \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \to H_{n-1}(A \cap B) \to \cdots,$$

review: $X = A \cup B$ is torus, $A \cap B = S^1 \sqcup S^1$ disjoint union, $H_n(A) = H_n(B) = \mathbb{Z}$ for $n = 0, 1$ and 0 otherwise, $H_n(A \cap B) = H_n(A) \oplus H_n(B) = \mathbb{Z} \oplus \mathbb{Z}$ for $n = 0, 1$ and 0 otherwise. We can therefore continue with this sequence:

$$H_2(A) \oplus H_2(B) \to H_2(X) \to H_1(A \cap B) \xrightarrow{f} H_1(A) \oplus H_1(B) \to H_1(X) \to$$

$$\to H_0(A \cap B) \xrightarrow{g} H_0(A) \oplus H_0(B) \to H_0(X) \to 0,$$

and we can rewrite it as

$$0 \to H_2(X) \to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{f} \mathbb{Z} \oplus \mathbb{Z} \to H_1(X) \to$$

$$\to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{g} \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to 0,$$

where we use the fact, that torus is connected, so $H_0(X) = \mathbb{Z}$. Now we want to compute $H_2(X)$ and $H_1(X)$.

We know $H_2(X)$ is $\ker f$, $\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$, $(a, b) \mapsto (a + b, a + b)$, then $(a, -a)$ is in the kernel, so $H_2(X) = \mathbb{Z}[a, -a] = \mathbb{Z}$.

For the $H_1(X)$ group use the fact, that $\ker g$ is $\mathbb{Z}$ (it has same idea, basically). Now consider the sequence

$$0 \to \mathbb{Z} \hookrightarrow H_1(X) \to \ker g = \mathbb{Z} \to 0,$$

which splits, so $H_1(X) = \mathbb{Z} \oplus \mathbb{Z}$. We are done.

Sphere with two handels might be a homework.

□