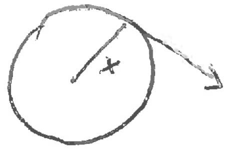
Exercise 1. *Prove that S^n has a nonzero vector field if and only if n is odd.*

Řešení

Nenulové vektorové pole na S^1

$$v(x_1, x_2) = (x_2, -x_1)$$


$v(x)$

Na S^2 nenulové vekt. pole neexistuje (věta o učesaném míči)

$v: S^n \longrightarrow \mathbb{R}^{n+1}$ $\quad v(x) \neq 0$ indukuje

$f: S^n \longrightarrow S^n$ $\quad f(x) = \frac{v(x)}{\|v(x)\|}$ $\quad f(x) \perp x$

Nechť takové zobrazení f existuje. Uvažujme homotopii $h \; S^n \times [0, \pi] \longrightarrow S^n$

$$h(x,t) = x \cos t + f(x) \sin t$$

$$\| x \cos t + f(x) \sin t \|^2 = \|x\|^2 \cos^2 t + \|f(x)\|^2 \sin^2 t = 1$$

$t = 0 \quad h(x, 0) = x \quad id_{S^n}$

$t = \pi \quad h(x, \pi) = -x \quad -id_{S^n}$

$\deg id_{S^n} = 1 = \deg(-id_S) = (-1)^{n+1} \Rightarrow n$ je liché

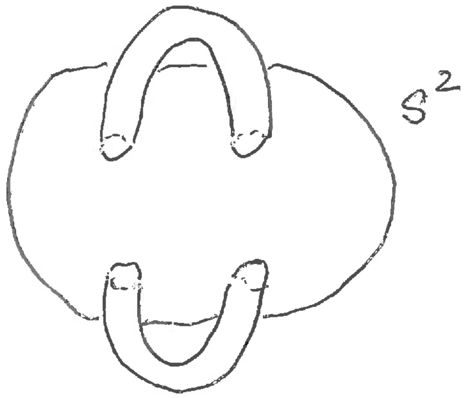
Pro n liché vekt. pole existuje $\quad S^{2n-1} \subseteq \mathbb{R}^{2n}$

$$v(x_1, \dots, x_{2n}) = (x_2, -x_1, x_4, -x_3, \dots, x_{2n}, -x_{2n-1})$$

$$\langle x, v(x) \rangle = x_1 x_2 - x_2 x_1 + x_3 x_4 - x_4 x_3 + \dots = 0.$$

Exercise 2. *Compute homology groups of oriented two dimensional surfaces using a suitable structure of CW-complex.*

Orientované kompaktní variety jsou až na homeomorfismus sféry s g ručkami.



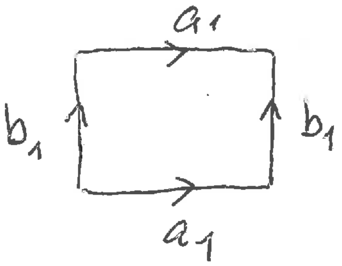
4 díry „zalepíme" po dvou ručkami

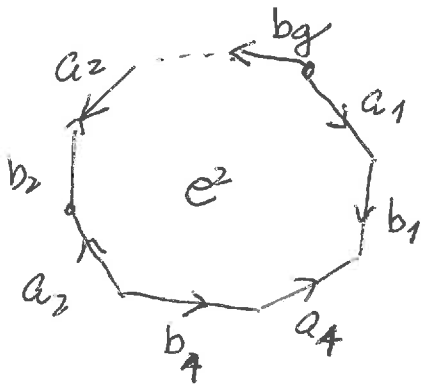
$S^1 \times [0,1]$

Má strukturu CW-komplexu s modelem

$$M_g = e^0 \cup e^1_1 \cup e^1_2 \cup \dots\dots e^1_{2g} \cup e^2$$

$g = 1$ torus





Hather

$$C_*^{CW}(M_g): \quad \underset{3}{0} \longrightarrow \underset{2}{\mathbb{Z}} \xrightarrow{\;0\;} \underset{1}{\bigoplus_{1}^{2g}} \mathbb{Z} \xrightarrow{\;0\;} \mathbb{Z} \longrightarrow 0$$

Proto

$$H_0(M_g) \cong \mathbb{Z}$$

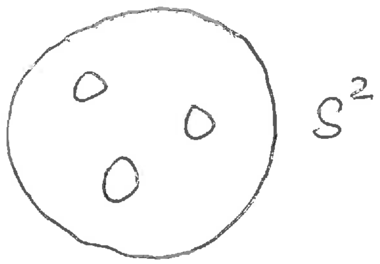
$$H_1(M_g) \cong \bigoplus_{1}^{2g} \mathbb{Z}$$

$$H_2(M_g) \cong \mathbb{Z}$$

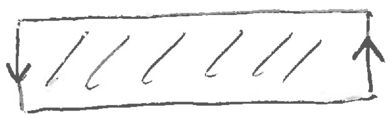
2(ii)

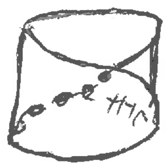
Neorientovatelné 2-dim kompaktní variety

N_g



S^2 s g vyřezanými disky, které zalepíme Möbiovými proužky





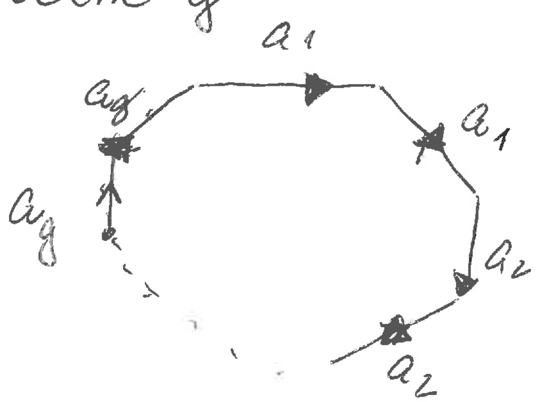
Model pro $g = 1$, projektivní prostor

$\partial e^2 = S^1$ $\qquad N_1 = e^0 \cup e^1_1 \cup e^2$

$f : \partial D^2 \longrightarrow S^1$ zobrazení stupně 2

$$N_1 = S^1 \cup_f D^2$$

Obecné g



$$\begin{array}{llllllll} & 3 & & 2 & & 1 & & 0 \\ C^{cw}(N_g) & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{d} & \bigoplus\limits_1^g \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} \xrightarrow{0} 0 \end{array}$$

$$d(e^2) = 2e^1_1 + 2e^1_2 + \quad + 2e^1_g$$

$$H_0(N_g) \cong \mathbb{Z}$$

$$H_2(N_g) \cong 0$$

$$H_1(N_g) \cong \frac{\bigoplus\limits_1^g \mathbb{Z}}{[(2,2,\dots 2)]} \cong \frac{\mathbb{Z}[e_1, \dots e_{g-1}, e_1 + e_2 + \dots + e_g]}{2\,\mathbb{Z}[e_1 + e_2 + \dots + e_g]}$$

$$\cong \bigoplus_1^{g-1} \mathbb{Z} \oplus \mathbb{Z}/2$$

Exercise 3. *Have $f\colon S^n \to S^n$ map of degree k. (such map always exists). Let $X = D^{n+1} \cup_f S^n$ and compute homology of X and the projection $p\colon X \to X/S^n$ in homology.*

$X = D^{n+1} \cup_f S^n$ je CW-komplex

$X = e^0 \cup e^n \cup e^{n+1}$

$$C^{CW}(X): \quad 0 \longrightarrow \underset{n+1}{\mathbb{Z}} \xrightarrow[k\cdot]{} \underset{n}{\mathbb{Z}} \longrightarrow \underset{n-1}{0} \cdots \longrightarrow \underset{0}{\mathbb{Z}}$$

$H_0(X) \cong \mathbb{Z}$

$H_n(X) \cong \mathbb{Z}/k\mathbb{Z} \cong \mathbb{Z}/k$

$H_{n+1}(X) \cong 0$

Projekce $p: X \longrightarrow X/S^n \cong S^{n+1}$

$$p_*: \underset{\substack{\| \\ 0}}{H_{n+1}(X)} \longrightarrow \underset{\substack{\| \\ \mathbb{Z}}}{H_{n+1}(S^{n+1})} \qquad p_* = 0$$

$$p_*\ \underset{\substack{\| \\ \mathbb{Z}}}{H_n(X)} \longrightarrow \underset{\substack{\| \\ 0}}{H_n(S^{n+1})} \qquad p_* = 0$$

Vezměme homologie s koeficienty v $\mathbb{Z}/k$

$H_{n+1}(X) \cong \mathbb{Z}/k \qquad H_n(X) \cong \mathbb{Z}/k$

V tomto případě $p_*\ H_{n+1}(X; \mathbb{Z}/k) \longrightarrow H_{n+1}(S^{n+1}; \mathbb{Z}/k)$ je identita, ~~ke~~ neboť generátor $[e^{n+1}]$ v $H_{n+1}(X; \mathbb{Z}/k)$ se zobrazí do $[e^{n+1}]$ generátoru v $H_{n+1}(S^{n+1}; \mathbb{Z}/k)$.

$\boxed{\text{p je homotopicky netriviální!}}$

Let X be a topological space with finitely generated homological groups and let $H_i(X) = 0$ for each sufficiently large i. Every finitely generated abelian group can be written as $\underbrace{\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{k-times} \oplus \mathrm{Tor}$, *where* Tor *denote torsion part of the group. The number k is called the* rank *of the group.*

Euler characteristic χ of X is defined by:

$$\chi(X) = \sum_{i=0}^{\infty} (-1)^i \operatorname{rank} H_i(X)$$

Example. *We know* $H_i(S^n) = \begin{cases} \mathbb{Z}, & i = 0, n \\ 0, & \textit{otherwise.} \end{cases}$ *Thus* $\chi(S^n) = 1 - (-1)^n$.

Exercise 4. *Let (C_*, ∂) be a chain complex with homology $H_*(C_*)$. Prove that $\chi(X) = \chi(C_*)$, where*

$$\chi(C_*) = \sum_{i=0}^{\infty} (-1)^i \operatorname{rank} C_i.$$

Aby měly definované výrazy smysl, předpokládáme, že všechny ranky jsou konečné a nenulové pouze pro konečně mnoho i.

$$H_i = \frac{Z_i}{B_i} \qquad \frac{\text{cykly } \ker \partial_i}{\text{hranice } \operatorname{im} \partial_{i+1}}$$

$$\operatorname{rank} H_i(C_*) = \operatorname{rank} Z_i - \operatorname{rank} B_i$$

Máme krátkou exaktní posloupnost

$$0 \to Z_i \hookrightarrow C_i \xrightarrow{\partial_i} B_{i-1} \to 0$$

Odtud $\operatorname{rank} C_i = \operatorname{rank} Z_i + \operatorname{rank} B_{i-1}$

$$\begin{aligned}
\chi(C_*) &= \operatorname{rank} C_0 - \operatorname{rank} C_1 + \operatorname{rank} C_2 - \operatorname{rank} C_3 + \cdots \\
&= \operatorname{rank} Z_0 - (\operatorname{rank} Z_1 + \operatorname{rank} B_0) + (\operatorname{rank} Z_2 + \operatorname{rank} B_1) \\
&\quad - (\operatorname{rank} Z_3 + \operatorname{rank} B_2) + \cdots \\
&= (\operatorname{rank} Z_0 - \operatorname{rank} B_0) - (\operatorname{rank} Z_1 - \operatorname{rank} B_1) \\
&\quad + (\operatorname{rank} Z_2 - \operatorname{rank} B_2) - \cdots \\
&= \operatorname{rank} H_0 - \operatorname{rank} H_1 + \operatorname{rank} H_2 - \cdots = \chi(H_*(C_*))
\end{aligned}$$

Let X be a topological space with finitely generated homological groups and let $H_i(X) = 0$ for every sufficiently large i. Let $f\colon X \to X$ be a continuous map. Map f induces homomorphism on the chain complex $f_*\colon C_*(X) \to C_*(X)$ and on the homologiy groups $H_*f\colon H_*(X) \to H_*(X)$, where $H_*f(\operatorname{Tor} H_*(X)) \subseteq \operatorname{Tor} H_*(X)$. Thus it induces homomorphism

$$H_*f\colon H_*(X)/\operatorname{Tor} H_*(X) \to H_*(X)/\operatorname{Tor} H_*(X).$$

Since $H_*(X)/\operatorname{Tor} H_*(X) \cong \underbrace{\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{\operatorname{rank} H_*(X)}$, map H_*f can be written as a matrix, thus we can compute its trace. So we can define the Lefschetz number of a map f:

$$L(f) = \sum_{i=0}^{\infty} (-1)^i \operatorname{tr} H_i f.$$

Similarly to the case of the Euler characteristic, it can be proved that[1]

$$\sum_{i=0}^{\infty} (-1)^i \operatorname{tr} H_i f = \sum_{i=0}^{\infty} (-1)^i \operatorname{tr} f_i.$$

Theorem. *If $L(f) \neq 0$, then f has a fixed point.*

Exercise 5. *Use the theorem above to show, that every continuous map f on D^n and $\mathbb{R}P^n$ where n is even has a fixed point.*

$f : D^n \to D^n$ $\qquad H_i(D^n) = \begin{cases} \mathbb{Z} & \text{pro } i=0 \\ 0 & \text{jinak} \end{cases}$

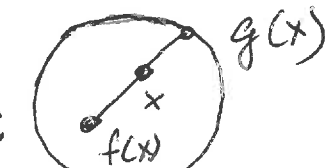
$H_0 f : \mathbb{Z} \to \mathbb{Z}$ je identita, proto $L(f) = 1$ a tedy $f: D^n \to D^n$ má pevný bod.

To je znění Browerovy věty.

Lze dokázat i jinak – například z toho, že S^{n-1} není retrakt D^n

$$\begin{tikzcd} S^{n-1} \arrow[r, hook, "i"] \arrow[dr, "id"'] & D^n \arrow[d, "r"] \\ & S^{n-1} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{Z} = H^{n-1}(S^{n-1}) \arrow[r] \arrow[dr, "id"'] & H^{n-1}(D^n) = 0 \arrow[d] \\ & H^{n-1}(D^n) = \mathbb{Z} \end{tikzcd}$$

Nyní: f $D^n \to D^n$ nemá pevný bod, pak g $D^n \to S^{n-1}$ je retrakce. Spor.



5 (ii)

$f : \mathbb{R}P^n \longrightarrow \mathbb{R}P^n$ n sudé

$$H_i(\mathbb{R}P^n) \cong \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z}/2 & i \text{ sude } \quad 0<i<n \\ 0 & \text{jinak} \end{cases}$$

$$H_i(\mathbb{R}P^n)\big/\mathrm{Tor} \cong \begin{cases} \mathbb{Z} & \text{pro } i=0 \\ 0 & \text{jinak} \end{cases}$$

$L(f) = \mathrm{tr}\,(id : \mathbb{Z} \to \mathbb{Z}) = 1$

f má pevný bod.

Pro n liché to není pravda

$$H_n(\mathbb{R}P^n) \cong \mathbb{Z}$$

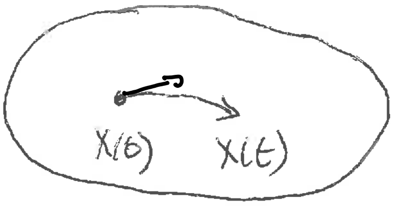
a $L(f)$ může být roven 0.

Exercise 6. *Let M be a smooth compact manifold. Prove implication the $\Rightarrow$ in the statement, that there is a nonzero* tečné *vector field on M if and only if $\chi(M) = 0$.*

Myslí se tečné vekt. pole. Nechť existuje nenulové tečné vett. pole. Pak řešíme rovnici

$$\dot{X}(t) = v(X(t)) \quad \text{na } M$$

$$X(0) = x$$



Z kompaktnosti a vlastnosti řešení existuje $t > 0$, že

$$X(0) \neq X(t)$$

$$X(0) = id_M \qquad X(t) = f \qquad \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

Zřejmě $id_M \sim f$

$$L(id_M) = L(f) = 0$$ neboť f nemá pevný bod

Platí $\operatorname{tr} id_{H(M)} = \operatorname{rank} H_i(M)$. Proto

$$0 = L(f) = L(id_M) = \chi(M).$$

Tedy nutná podmínka je, že $\chi(M) = 0$.