Exercise 1. *Prove that S^n has a nonzero vector field if and only if n is odd.*

Solution. First note, that we have $v\colon S^n \to \mathbb{R}^{n+1}$ such that $v(x) \perp x$. Consider the case of S^1 and $(x_0, x_1) \mapsto (x_1, -x_0)$. Take $(x_0, x_1, x_2, x_3, \ldots, x_{2n+1}) \in S^{2n+1} \subseteq \mathbb{R}^{2n}$. Then we get $(x_1, -x_0, x_3, -x_2, \ldots, x_{2n+1}, -x_{2n})$ as image and there is nothing more obvious than that the product is zero, i.e. it's perpendicular.

Note, for $S^1 \subseteq \mathbb{C}$ it is $z \mapsto ez$, where e is the complex unit, usually denoted as i.

Now we want to prove that if S^n has a nonzero vector field, then n is odd. We use the fact that $\deg(\mathrm{id}) = 1$ and $\deg(-\mathrm{id}) = (-1)^{n+1}$. Take $v\colon S^n \to S^n$. If we show $\mathrm{id} \sim -\mathrm{id}$, then $1 = (-1)^{n+1} \Rightarrow n$ is odd. The homotopy is $h(x,t)$ we are looking for is $h(x,t) = cos(t)x + sin(t)v(x)$, where $t \in [0, \pi]$. Also note $\parallel h(x,t) \parallel = cos^2 t + sin^2 t = 1$. We are done. □

Exercise 2. *Compute homology groups of oriented two dimensional surfaces using a suitable structure of CW-complex.*

Solution. Denote M_g surface of genus g (it is the same as sphere with g handles, i.e. M_1 is torus and M_2 is double torus (homework 4). The CW-model is $e^0 \cup e^1_1 \cup \cdots \cup e^1_{2g} \cup e^2$ and we have

$$0 \to \mathbb{Z} \to \bigoplus_{1}^{2g} \mathbb{Z} \to \mathbb{Z} \to 0.$$

Second differential is zero, $e^0 - e^0 = 0$. The first one is zero as well (glue the model, the arrows go with $+$ and then $-$). Then we get $H_0 = \mathbb{Z}, H_1 = \bigoplus_1^{2g} \mathbb{Z}, H_2 = \mathbb{Z}$.

For nonorientable surfaces, N_g is modelled by one 2-dimensional disc which has boundary composed with g segments every of which repeats twice with the same orientation. So we have one cell in dimensions 2 and 0 and g cells in dimension one. We get (quite similarly)

$$0 \to \mathbb{Z} \to \bigoplus_{1}^{g} \mathbb{Z} \to \mathbb{Z} \to 0.$$

This equality holds: $d[e^2] = 2[e^1_1] + 2[e^1_2] + \ldots$, so $H_2 = 0, H_0 = \mathbb{Z}$ and the only interesting case is

$$H_1 = \frac{\mathbb{Z}[e^1_1, \ldots, e^1_g]}{\mathbb{Z}[2e^1_1 + \cdots + 2e^1_g]} = \mathbb{Z}_2 + \bigoplus_{1}^{g-1} \mathbb{Z}.$$

□

Exercise 3. *Have $f\colon S^n \to S^n$ map of degree k. (such map always exists). Let $X = D^{n+1} \cup_f S^n$ and compute homology of X and the projection $p\colon X \to X/S^n$ in homology.*

Solution. Easy, $X = e^0 \cup e^n \cup e^{n+1}$ and $0 \to \mathbb{Z} \xrightarrow{k\cdot} \mathbb{Z} \to 0 \to \cdots \to 0 \to \mathbb{Z} \to 0$, also

$$S^n \to X^{(n)} \to X^{(n)}/(X^{(n)} - e^n) = S^n.$$

We get $H_{n+1}(X) = 0, H_n = \mathbb{Z}_k, H_0(X) = \mathbb{Z}$. Note, $X/S^n \cong S^{n+1}$. So, for the p_* we have

$$p_*\colon H_{n+1}(X) = 0 \xrightarrow{0} H_{n+1}(S^{n+1}) = \mathbb{Z}$$

and

$$p_*\colon H_n(X) = \mathbb{Z}_k \xrightarrow{0} H_n(S^{n+1}) = 0$$

and at H_0 it is identity $\mathbb{Z} \to \mathbb{Z}$.

□

Let X be a topological space with finitely generated homological groups and let $H_i(X) = 0$ for each sufficiently large i. Every finitely generated abelian group can be written as $\underbrace{\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{k-times} \oplus \operatorname{Tor}$, where Tor *denote torsion part of the group. The number k is called the* rank *of the group.*

Euler characteristic χ of X is defined by:

$$\chi(X) = \sum_{i=0}^{\infty} (-1)^i \operatorname{rank} H_i(X)$$

Example. *We know* $H_i(S^n) = \begin{cases} \mathbb{Z}, & i = 0, n \\ 0, & \textit{otherwise.} \end{cases}$ *Thus* $\chi(S^n) = 1 - (-1)^n$.

Exercise 4. *Let (C_*, ∂) be a chain complex with homology $H_*(C_*)$. Prove that $\chi(X) = \chi(C_*)$, where*

$$\chi(C_*) = \sum_{i=0}^{\infty} (-1)^i \operatorname{rank} C_i.$$

Solution. We have two short exact sequences:

$$0 \to Z_i \hookrightarrow C_i \xrightarrow{\partial} B_{i-1} \to 0$$
$$0 \to B_i \hookrightarrow Z_i \to Z_i/B_i = H_i \to 0,$$

where C_i, cycles Z_i and boundaries B_i are free abelian groups, thus $\operatorname{rank} C_i = \operatorname{rank} Z_i + \operatorname{rank} B_{i-1}$ and $\operatorname{rank} H_i = \operatorname{rank} Z_i - \operatorname{rank} B_i$. Thus we have

$$\begin{aligned} \chi(C_*) &= \sum_{i=0}^{\infty} (-1)^i \operatorname{rank} Z_i + \sum_{i=0}^{\infty} (-1)^i \operatorname{rank} B_{i-1} \\ &= \sum_{i=0}^{\infty} (-1)^i \operatorname{rank} Z_i - \sum_{i=0}^{\infty} (-1)^i \operatorname{rank} B_i = \chi(X). \end{aligned}$$

□

Let X be a topological space with finitely generated homological groups and let $H_i(X) = 0$ for every sufficiently large i. Let $f\colon X \to X$ be a continuous map. Map f induces homomorphism on the chain complex $f_*\colon C_*(X) \to C_*(X)$ and on the homologiy groups

$H_*f\colon H_*(X) \to H_*(X)$, where $H_*f(\operatorname{Tor} H_*(X)) \subseteq \operatorname{Tor} H_*(X)$. Thus it induces homomorphism

$$H_*f\colon H_*(X)/\operatorname{Tor} H_*(X) \to H_*(X)/\operatorname{Tor} H_*(X).$$

Since $H_*(X)/\operatorname{Tor} H_*(X) \cong \underbrace{\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{\operatorname{rank} H_*(X)}$, map H_*f can be written as a matrix, thus we can compute its trace. So we can define the Lefschetz number of a map f:

$$L(f) = \sum_{i=0}^{\infty} (-1)^i \operatorname{tr} H_i f.$$

Similarly to the case of the Euler characteristic, it can be proved that[1]

$$\sum_{i=0}^{\infty} (-1)^i \operatorname{tr} H_i f = \sum_{i=0}^{\infty} (-1)^i \operatorname{tr} f_i.$$

Theorem. *If $L(f) \neq 0$, then f has a fixed point.*

Exercise 5. *Use the theorem above to show, that every cts map f on D^n and $\mathbb{R}P^n$ where n is even has a fixed point.*

Solution. We know that that $H_i(D^n) = \begin{cases} \mathbb{Z}, & i = 0 \\ 0, & \text{otherwise.} \end{cases}$ Because $H_0 f\colon H_0(D^n) \cong \mathbb{Z} \to \mathbb{Z} \cong H_0(D^n)$ can be only the identity, we have $L(f) = 1$, thus f has a fixed point.

Since $H_i(\mathbb{R}P^n) = \begin{cases} \mathbb{Z}, & i = 0, \\ \mathbb{Z}/2, & i < n, i \text{ odd}; \\ 0, & \text{otherwise,} \end{cases}$ and $\mathbb{Z}/2$ is torsion, we have $L(f) = 1$ as in the previous case. □

Exercise 6. *Let M be a smooth compact manifold. Prove, that there is a nonzero vector field on M if and only if $\chi(M) = 0$.*

Solution. We will prove only implication ⇒. Let v be a nonzero vector field on M. Define a map $X\colon [0,1] \times M \to M$ which satisfies $\dot{X}(t,x) = v(X(t,x))$ for every $x \in M$ and $X(0,x) = x$. There exists t_0 such that $X(t_0,x) \neq x$. Denote $f(x) = X(t_0,x)$, thus f has no fixed point, thus $L(f) = 0$. Because f is homotopic to id and $\operatorname{tr} H_i \mathrm{id} = \operatorname{rank} H_i(M)$, we get from homotopy invariance $0 = L(f) = L(\mathrm{id}) = \chi(M)$. □

[1] $f_i\colon C_i(X) \to C_i(X)$