Exercise 1. *Use $\mathbb{Z}/2$ coefficients to show, that every cts map $f\colon S^n \to S^n$ satisfying $f(-x) = -f(x)$ has an odd degree.*

Solution. The map f induces a map $g\colon \mathbb{R}P^n \to \mathbb{R}P^n$, since $f(\{x,-x\}) \subseteq \{f(x), -f(x)\}$. We have the short exact sequence[1]

$$\begin{array}{ccccccccc} & & \sigma & \longmapsto & \sigma_1 + \sigma_2 & \longmapsto & 2\sigma = 0 & & \\ 0 & \to & C_*(\mathbb{R}P^n, \mathbb{Z}/2) & \to & C_*(S^n, \mathbb{Z}/2) & \to & C_*(\mathbb{R}P^n, \mathbb{Z}/2) & \to & 0, \end{array}$$

where $\sigma\colon \Delta^i \to \mathbb{R}P^n$ is an arbitrary element of $C_*(\mathbb{R}P^n)$, σ_1, σ_2 are its preimages of a projection:

$$\begin{array}{ccc} & \overset{\sigma_1,\sigma_2}{\nearrow} & S^n \\ & & \downarrow \\ \Delta^i & \xrightarrow[\sigma]{} & \mathbb{R}P^n \end{array}$$

From the short exact sequence we get the long exact sequence

$$\begin{array}{ccccccccc} H_i(\mathbb{R}P^n;\mathbb{Z}/2) & \to & H_i(S^n;\mathbb{Z}/2) & \to & H_i(\mathbb{R}P^n;\mathbb{Z}/2) & \to & H_{i-1}(\mathbb{R}P^n;\mathbb{Z}/2) & \to & 0 \\ \downarrow g_* & & \downarrow f_* & & \downarrow g_* & & \downarrow g_* & & \\ H_i(\mathbb{R}P^n;\mathbb{Z}/2) & \to & H_i(S^n;\mathbb{Z}/2) & \to & H_i(\mathbb{R}P^n;\mathbb{Z}/2) & \to & H_{i-1}(\mathbb{R}P^n;\mathbb{Z}/2) & \to & 0 \end{array}$$

Because $H_0(\mathbb{R}P^n;\mathbb{Z}/2) = \mathbb{Z}/2$ and g_0 on $H_0(\mathbb{R}P^n;\mathbb{Z}/2)$ is an isomorphism, we can show by induction, that $H_i(\mathbb{R}P^n, \mathbb{Z}/2) = \mathbb{Z}/2$ and g_i is an isomorphism for every $i \leq n-1$. An induction step is shown on the following diagram (three isomorphisms imply the fourth):

$$\begin{array}{ccccc} 0 & \to & \mathbb{Z}/2 & \xrightarrow{\cong} & \mathbb{Z}/2 \\ & & \downarrow \cong & & \downarrow \cong \\ 0 & \to & \mathbb{Z}/2 & \xrightarrow[\cong]{} & \mathbb{Z}/2 \end{array}$$

For $i = n$ we have the following situation (the vertical isomorphisms were proved by induction):

$$\begin{array}{ccccccccc} \mathbb{Z}/2 & \xrightarrow{\cong} & \mathbb{Z}/2 & \xrightarrow{0} & \mathbb{Z}/2 & \xrightarrow{\cong} & \mathbb{Z}/2 & \to & 0 \\ \downarrow \cong & & \downarrow ? & & \downarrow \cong & & \downarrow \cong & & \\ \mathbb{Z}/2 & \xrightarrow[\cong]{} & \mathbb{Z}/2 & \xrightarrow{0} & \mathbb{Z}/2 & \xrightarrow[\cong]{} & \mathbb{Z}/2 & \to & 0 \end{array}$$

Thus f_* (the arrow marked by ?) has to be an isomorphism for H_n, thus it maps $[1]_2$ to $[1]_2$, hence f has degree $1 \mod 2$. □

Exercise 2. *Let $\varphi \in C^k(X;R), \psi \in C^l(Y;R)$. Prove $\delta(\varphi \cup \psi) = \delta\varphi \cup \psi + (-1)^k \varphi \cup \delta\psi$. Use $\tau = [e_0, \ldots, e_{k+l+1}] \in C_{k+l+1}(X)$.*

[1] $2\sigma = 0$ because of the $\mathbb{Z}/2$ coefficient.

Solution. Easily work out

$$\delta(\varphi \cup \psi)(\tau) = (\varphi \cup \psi)(\delta\tau) = (\varphi \cup \psi)\Big(\sum_{i=0}^{k+l+1} (-1)^i \tau/[e_0, \dots, \hat{e_i}, \dots, e_{k+l+1}] \Big) =$$

$$= \sum_{i=0}^{k} (-1)^i \varphi(\tau/[e_0, \dots, \hat{e_i}, \dots, e_{k+1}])\psi(\tau/[e_{k+1}, \dots, e_{k+l+1}]) +$$
$$+ \sum_{i=k+1}^{k+l+1} (-1)^i \varphi(\tau/[e_0, \dots, e_k])\psi(\tau/[e_k, \dots, \hat{e_i}, \dots, e_{k+l+1}]).$$

Now, the right hand side of the formula, the first part gives

$$\begin{aligned}(\delta\varphi \cup \psi)(\tau) &= \delta\varphi(\tau/[e_0, \dots, e_{k+1}]) \cdot \psi(\tau/[e_{k+1}, \dots, e_{k+l+1}]) = \\ &= \varphi(\delta\tau/[e_0, \dots, e_{k+1}]) \cdot \psi(\tau/[e_{k+1}, \dots, e_{k+l+1}]) = \\ &= \sum_{i=0}^{k+1} (-1)^i \varphi(\delta\tau/[e_0, \dots, \hat{e_i}, \dots, e_{k+1}]) \cdot \psi(\tau/[e_{k+1}, \dots, e_{k+l+1}]).\end{aligned}$$

The second part is

$$\begin{aligned}(-1)^k(\varphi \cup \delta\psi)(\tau) &= (-1)^k \varphi(\tau/[e_0, \dots, e_k])\delta\psi(\tau/[e_k, \dots, e_{k+l+1}]) = \\ &= \sum_{j=0}^{l+1} (-1)^{j+k} \varphi(\delta\tau/[e_0, \dots, e_k]) \cdot \psi(\tau/[e_k, \dots, \hat{e}_{k+j}, \dots, e_{k+l+1}]).\end{aligned}$$

Now, the last summand of the first part plus the first summand of the second part yields

$$(-1)^{k+1}\varphi(\tau/[e_0, \dots, e_k])\psi(\tau/[e_{k+1}, \dots, e_{k+l+1}]) +$$
$$+ (-1)^k \varphi(\tau/[e_0, \dots, e_k])\psi(\tau/[e_{k+1}, \dots, e_{k+l+1}]) = 0,$$

and we are done, $\mathsf{LHS} = \mathsf{RHS}$.

□

Exercise 3. *Compute the structure of graded algebra $H^*(S^n \times S^n; \mathbb{Z})$ for n even and n odd. Use the following:*

If $H^n(Y; R)$ is free finitely generated group for all n and $(X, A), Y$ are CW-complexes, then

$$\times \colon H^*(X, A; R) \otimes H^*(Y; R) \to H^*(X \times Y, A \times Y; R)$$

is an isomorphism of graded rings.

Solution. We will omit writing the $\mathbb{Z}$ coefficients.

Now, $H^*(S^n) \otimes H^*(S^n) \to H^*(S^n \times S^n)$ and we know that for spheres $H^0 = \mathbb{Z}$ with generator 1 and $H^n = \mathbb{Z}$, denote generator a. Also, $a \cup a \in H^{2n} = 0, a \cup a = 0$, so we get $\mathbb{Z}[a]/\langle a^2\rangle$ and $\deg(a) = n$. We can write the same for the second, so denote the other generator b and have $\deg(b) = n$ and we have $\mathbb{Z}[b]/\langle b^2\rangle$.

Now we compute tensor product $\mathbb{Z}[a]/\langle a^2\rangle \otimes \mathbb{Z}[b]/\langle b^2\rangle$, we have four generators: $1_a \otimes 1_b, a \otimes 1_b, 1_a \otimes b, a \otimes b$, we will denote them $1, c, d, c \cdot d$. Compute

$$(a \otimes 1_b) \cdot (1_a \otimes b) = (-1)^{0\cdot 0}(a \cdot 1_a) \otimes (1_b \cdot b) = a \otimes b,$$

because 0 is an idempotent element, i.e. $0\cdot 0 = 0$, and $(-1)^n = 1$ for n even, again, as in the first exercise, we use Evenness of Zero. (We refer the reader to "Principia Mathematica" *Whitehead, Russell*,(1910,1912,1913).) Continue with computation

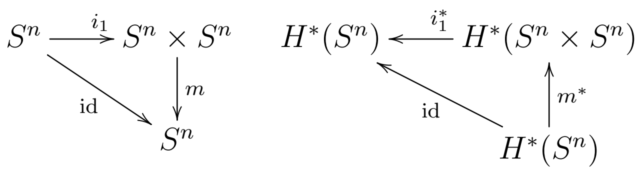
$$(1_a \otimes b) \cdot (a \otimes 1_b) = (-1)^{n\cdot n}(1_a \cdot a) \otimes (b \cdot 1_b) = (-1)^n a \otimes b,$$

so the algebra we get is $H^*(S^n \times S^n) = \mathbb{Z}[c, d]/\langle c^2, d^2, dc - (-1)^n cd\rangle$. For n even we have $dc = cd$. □

Exercise 4. *Prove that there is no multiplication on even dimensional spheres. Multiplication on the sphere S^n is a map $m\colon S^n \times S^n \to S^n$ such that there is an element $1 \in S^n$ satisfying $m(x, 1) = x, m(1, x) = x$.*

Hint: compute $m^*\colon H^*(S^n) \to H^*(S^n \times S^n)$, describe two rings.

Solution. We have $H^*(S^n) = \mathbb{Z}[\gamma]/\langle\gamma^2\rangle$ and $H^*(S^n \times S^n) = \mathbb{Z}[\alpha, \beta]/\langle\alpha^2, \beta^2\rangle = H^*(S^n) \otimes H^*(S^n)$, because we already know, that $\alpha\beta = \beta\alpha$. Our situation can be described with two diagrams:

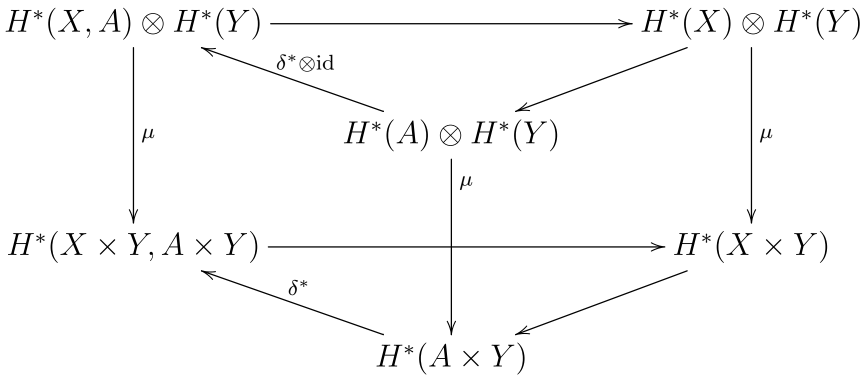


Take $m^*(\gamma) = a\alpha + b\beta$ with $a, b \in \mathbb{Z}$ and prove first, that $a = b = 1$. Use $m \circ i_1 = \mathrm{id}$, so $i_1^*(m^*\gamma) = \gamma$. This gives $i_1^*(a\alpha + b\beta) = \gamma$ and since $i_1^*(a\alpha + b\beta) = a\gamma$, we have $a = 1$, same for b. Final computation yields

$$\begin{aligned}0 &= m^*(0) = m^*(\gamma^2) = m^*(\gamma \cup \gamma) = m^*\gamma \cup m^*\gamma = \\ &= (\alpha + \beta) \cup (\alpha + \beta) = \alpha^2 + \alpha\beta + \beta\alpha + \beta^2 = 0 + 2\alpha\beta + 0 \neq 0,\end{aligned}$$

and that, my friends, is a contradiction. □

Exercise 5. *Show commutativity of the diagram below. Use five lemma to prove that taking any two μ's iso's, the third μ is iso as well.*



Solution. We will name the parts of the diagram as follows: back-square, upper-triangle, lower-triangle, left-square, right-square. The triangles come from the long exact sequence of of pairs (X, A) and $(X \times Y, A \times Y)$, the right-square commutativity comes from an inclusion, back-square commutes as well (topology knowledge). The only problematic part is the left-square and it is exercise on computation of connecting homomorphism δ^*. □