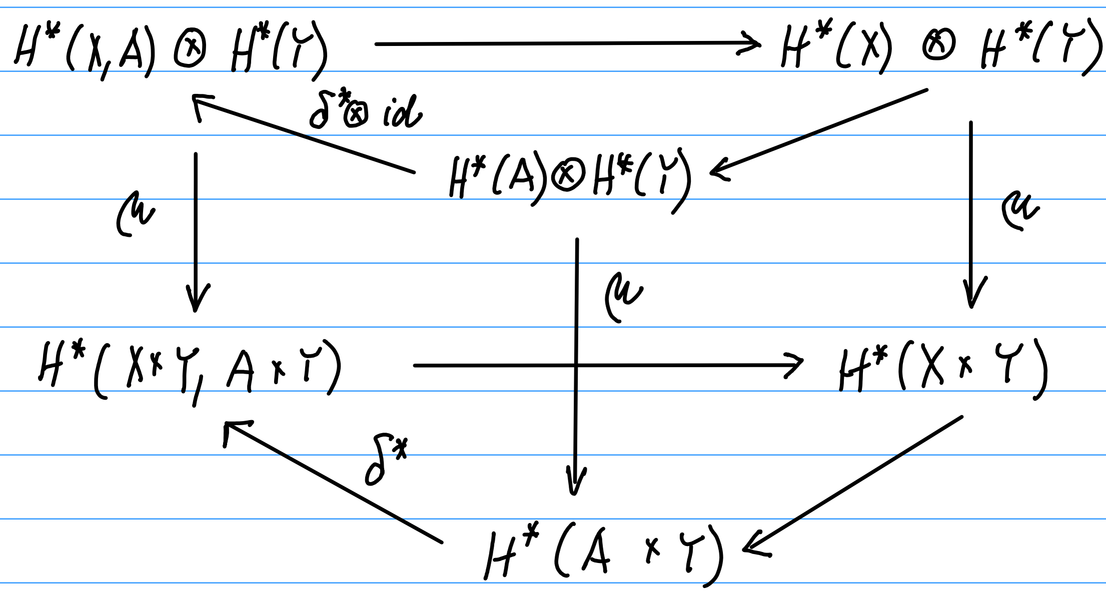
Lecture 7 Products in cohomology, Poincaré duality

Theorem: Let (X,A) and (Y,B) be pairs of CW-complexes. Suppose that $H^k(Y,B)$ are free finitely generated R-modules for all k. Then

$$\mu : H^*(X,A;R) \otimes H^*(Y,B;R) \longrightarrow H^*(X\times Y, X\times B \cup A\times Y; R)$$

defined as a cross product, is an isomorphism of graded rings.

Proof: In the tutorial we have already proved commutativity of the diagram



(1) We prove the statement for finite dimensional CW-complex X and $A = B = \emptyset$ by induction with respect to dimension of X.
Then holds for $\dim X = 0$. Suppose it holds for $\dim X = n-1$.

Apply the diagram for $X = X^n$, $A = X^{n-1}$ and Y.
We need to show that
$$\mu : H^*(X^n, X^{n-1}) \otimes H^*(Y) \longrightarrow H^*(X^n \times Y, X^{n-1} \times Y)$$
is an iso of Abelian groups. Then we can use 5-lemma to prove the statement of Thm for $X = X^n$.

Since $H^*(X^n, X^{n-1}) \cong \tilde{H}^*(X^n / X^{n-1}) \cong \tilde{H}^*(\bigvee_\alpha S^n_\alpha)$
and $H^*(X^n \times Y, X^{n-1} \times Y) \cong \tilde{H}(X^n \times Y / X^{n-1} \times Y) \cong \tilde{H}^*(\bigvee_\alpha S^n_\alpha \times Y)$,
it suffices to prove that
$$\mu \quad H^*(\bigvee S^n_\alpha) \otimes H^*(Y) \longrightarrow H^*(\bigvee_\alpha S^n_\alpha \times Y)$$
is an iso. For this we use above diagram for $X = \sqcup D^n_\alpha$, $A = \sqcup \partial D^n_\alpha$ and induction.

(2) X finite dimensional CW-complex, $A \subset X$ a subcomplex. We use the diagram above and 5-lemma to prove thm for (X, A).

(3) For X infinite CW-complex, we have to prove that $H^i(X) \cong H^i(X^n)$ for $i < n$ which is equivalent to $H^i(X / X^n) = 0$. See Hatcher pp. 220-221.

Eilenberg - Zilber Theorem

There are chain homomorphisms
$$C_*(X \times Y) \underset{\psi = EZ}{\overset{\varphi}{\rightleftarrows}} C_*(X) \otimes C_*(Y)$$
such that $\varphi(\sigma_0 \otimes \tau_0) = (\sigma_0, \tau_0)$ and $\psi(\sigma_0, \tau_0) = (\sigma_0 \times \tau_0)$, which are natural chain equivalences, i.e
$$\psi \circ \varphi \sim id_{C_*(X) \otimes C_*(Y)} \qquad \varphi \circ \psi \sim id_{C_*(X \times Y)}.$$

From this we get

$$H_*(X \times Y) \cong H_*(C_*(X) \otimes C_*(Y))$$

and

$$H^*(X \times Y) \cong H^*(C_*(X) \otimes C_*(Y))$$

but <u>not</u>

$$H_*(X \times Y) \cong H_*(X) \otimes H_*(Y)$$

or

$$H^*(X \times Y) \cong H^*(X) \otimes H^*(Y)$$

POINCARÉ DUALITY

Manifold M of dimension n – Hausdorff space where every point has a neighbourhood homeomorphic to $\mathbb{R}^n$.

We are going to define the orientation of a manifold using local homology groups.

For $x \in M$ and its neighbourhood U homeomorphic to $\mathbb{R}^n$

$$H_i(M, M \setminus x; \mathbb{Z}) \cong H_i(U, U \setminus x; \mathbb{Z}) \cong H_i(\mathbb{R}^n, \mathbb{R} \setminus 0; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & i = n \\ 0 & \text{otherwise} \end{cases}$$

Orientation in the point $x \in M$ is a choice of one from the two generators of

$$H_n(M, M \setminus x; \mathbb{Z}) \cong \mathbb{Z}.$$

We will denote this generator as μ_x.

Orientation of a manifold M

To every $x \in M$ we assign μ_x in such a way that the following compatibility condition holds.

For every $x \in M$ there is a neighbourhood B of x and an element $\mu_B \in H_n(M, M \setminus B; \mathbb{Z})$ induces the orientation μ_y for all $y \in B$.

$$\rho_y : (M, M \setminus B) \hookrightarrow (M, M - y)$$

$$\rho_{y*}(\mu_B) = \mu_y$$

Orientable manifold – enables a choice of compatible local orientations

Oriented manifold – a manifold with a choice of orientation

We can also define an orientation for coefficient in a group R .... the choice of generators in $H_n(M, M\setminus x; R) \cong R$.

We will use orientation with respect to $\mathbb{Z}$ and $\mathbb{Z}/2$. Since $H_n(M, M\setminus x; \mathbb{Z}/2) \cong \mathbb{Z}/2$ has only one generator, all manifolds are $\mathbb{Z}/2$-oriented.

It is not true for $\mathbb{Z}$-orientation.

Simply connected manifolds (every closed curve is contractible in M to a point) are orientable.

Fundamental class of a manifold M is a class $\mu \in H_n(M; \mathbb{Z})$ such that

$$(\rho_x)_*(\mu) = \mu_x \qquad \rho_x : M \hookrightarrow (M, M\setminus x)$$

Notation: The fundamental class is denoted as $[M]$.

Theorem: Let M be an orientable closed manifold. Then

(a) $H_i(M; \mathbb{Z}) \cong 0$ for all $i > n$.

(b) The natural map $H_n(M; \mathbb{Z}) \longrightarrow H_n(M, M\setminus x; \mathbb{Z})$ is an isomorphism for all $x \in M$.

The consequence of (b) is that every closed oriented manifold M has a fundamental class $[M] \in H_n(M;\mathbb{Z}) \cong \mathbb{Z}$.

Lemma: Let M be a manifold of dimension n and let $A \subseteq M$ be compact. Then
- $H_i(M, M \setminus A; \mathbb{Z}) = 0$ for $i > n$
- $\alpha \in H_i(M, M \setminus A; \mathbb{Z})$ is zero if and only if $(\rho_x)_*(\alpha) = 0$ for all $x \in A$.
- If $\{\mu_x\}$ is an orientation of M, then there is $\mu_A \in H_n(M, M\setminus A; \mathbb{Z})$ such that
$$(\rho_x)_*(\mu_A) = \mu_x .$$

We get immediately (a) from Theorem by applying the first point and $H_i(M, M\setminus x; \mathbb{Z}) \cong 0$
(b) of Thm follows from the second point if we take $A = M$.

Proof of Lemma:

(1) The statements hold for $A, B, A \cap B \subseteq M$. We will prove it for $A \cup B$ using Mayer-Vietoris Thm.

$$H_i(M, \underbrace{M\setminus(A\cup B)}_{(M\setminus A)\cap(M\setminus B)}) \longrightarrow H_i(M, M\setminus A) \oplus H_i(M, M\setminus B) \longrightarrow H_i(M, \underbrace{M\setminus(A\cap B)}_{(M\setminus A)\cup(M\setminus B)})$$

We get $H_i(M, M\setminus(A\cup B)) = 0$ for $i > n$

For $i = n$ $\qquad (\mu_A, \mu_B) \longmapsto \mu_{A\cap B} - \mu_{A\cap B} = 0$

$$0 \to H_n(M, M\setminus(A\cup B)) \to H_n(M, M\setminus A) \oplus H_n(M, M\setminus B) \to H_i(M, M\setminus(A\cap B))$$

Take μ_A, μ_B such that their restrictions are orientations of μ_x, then their images in $H_i(M, M\setminus(A\cap B))$ are the same and so there is $\mu_{A\cup B}$ and it is unique!

(2) $M = \mathbb{R}^n$, A a compact convex set. Then

$$H_i(\mathbb{R}^n, \mathbb{R}^n \setminus A) \cong H_i(\mathbb{R}^n, \mathbb{R}^n \setminus 0)$$

and we get the statements of lemma immediately.

(3) $M = \mathbb{R}^n$, $A = \bigcup_{i=1}^{k} A_i$ where A_i are compact convex sets. We can use (2) and (1) to get lemma in this case.

(4) $M = \mathbb{R}^n$, A a compact subset. Take $\alpha \in H_i(\mathbb{R}^n, \mathbb{R}^n \setminus A)$. This class is represented by a cycle $z \in C_i(\mathbb{R}^n)$ with boundary $\partial z \in C_{i-1}(\mathbb{R}^n \setminus A)$. Let C be the union of images of singular simplices in ∂z. Since C and A are compact and $C \cap A = \emptyset$, we get that

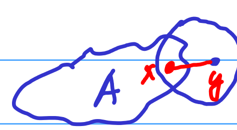
$$\operatorname{dist}(C, A) > 0.$$

So there is a union K of a finite number of balls (compact, convex) such that: $C \subset \mathbb{R}^n \setminus K$, $K \supset A$.

We have $\alpha_K \in H_i(\mathbb{R}^n, \mathbb{R}^n \setminus K)$ represented by the cycle $z \in C_i(\mathbb{R}^n)$ and

$$\alpha_K \longmapsto \alpha \quad \text{in} \quad H_i(\mathbb{R}^n, \mathbb{R}^n \setminus K) \to H_i(\mathbb{R}^n, \mathbb{R}^n \setminus A)$$

If $\alpha_K = 0$, then $\alpha = 0$. If $(\rho_x)_* \alpha = 0$ for $x \in A$, then also $(\rho_y)_* \alpha_K = 0$ for all $y \in K$ since there is a segment connecting y with a point x in A.



According to (3) $\alpha_K = 0$ and hence $\alpha = 0$.

(5) M a manifold, A compact set in $U \cong \mathbb{R}^n$. We have (using excision)

$$H_i(M, M \setminus A) \cong H_i(U, U \setminus A) \cong H_i(\mathbb{R}^n, \mathbb{R}^n \setminus A)$$

and we can use (4).

(6) M a manifold, $A \subseteq M$ compact.

Then $A \subseteq \bigcup_{i=1}^{k} U_i$, where $\overline{U}_i \subset V_i$ open homeomorphic to $\mathbb{R}^n$. Then $A = \bigcup_{i=1}^{k} (A \cap \overline{U}_i)$ and we can apply (5) to M, $A \cap \overline{U}_i$, (1) and induction.