Lecture 10 : Fundamental group

Covering space. A covering space of a space X is a space $\tilde{X}$ together with a map $p : \tilde{X} \to X$ such that $(\tilde{X}, X, p)$ is a fibre bundle with a discrete fibre.

Any fibre bundle has HLP. In the case of covering space the lift of a homotopy is unique.

Proposition Let $p : \tilde{X} \to X$ be a covering space and let Y be a space. Given a homotopy $F : Y \times I \to X$ and $\tilde{f} : Y \to \tilde{X}$ such that the square in the diagram commutes

$$\begin{tikzcd} Y \times \{0\} \arrow[r, "\tilde{f}"] \arrow[d, hook] & \tilde{X} \arrow[d, "p"] \\ Y \times I \arrow[r, "F"'] \arrow[ur, dashed, "\exists!"] & X \end{tikzcd}$$

there is just one lift $\tilde{F} : Y \times I \to \tilde{X}$ making both triangles commutative.

Corollary: $p_* \; \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ is injective.

Group actions: A left action of a discrete group G on a space Y is a map

$$G \times Y \to Y \qquad (g, y) \mapsto g \cdot y$$

such that $1 \cdot y = y$ and $(g_1 g_2) \cdot y = g_1 \cdot (g_2 \cdot y)$.

Properly discontinuous action: for each $y \in Y$ there is a neighbourhood U such that $g_1 U \cap g_2 U \neq \emptyset$ implies $g_1 = g_2$.

Orbit space Y/G is a quotient of Y given by the equivalence $x \sim y$ if $y = g \cdot x$.

A space Y is simply connected if it is path connected and $\pi_1(Y, y_0)$ is trivial for some (and hence all $y_0 \in Y$).

Theorem Let Y be a path connected space with a properly discontinuous action of a group G. Then
(1) The natural projection
$$p : Y \longrightarrow Y/G$$
is a covering space
(2)
$$G \cong \frac{\pi_1(Y/G, p(y_0))}{p_*(\pi_1(Y, y_0))}$$

Especially: If Y is simply connected
$$\pi_1(Y/G, p(y_0)) \cong G$$

Proof: Let $y \in Y$ and let U be the neighbourhood from the definition of properly discontinuous action. Then
$$p^{-1}(p(U)) = \text{disjoint union of } gU \text{ for all } g \in G$$
hence
$$p^{-1}(p(U)) \cong U \times G$$

So $(Y, Y/G, p)$ is a fibre bundle with the fibre G.

Applying the long exact sequence of a fibration we get

$$0 = \pi_1(G,1) \to \pi_1(Y, y_0) \xrightarrow{p_*} \pi_1(Y/G, p(y_0)) \xrightarrow{\partial} \pi_0(G) = G \to \pi_0(Y) = 0$$

One can show that ∂ is a group homomorphism. Consequently, the exact sequence implies

$$G \cong \frac{\pi_1(Y/G, p(y_0))}{p_*(\pi_1(Y, y_0))}$$

Example A $\mathbb{Z}$ acts on $\mathbb{R}$ by addition. The orbit space $\mathbb{R}/\mathbb{Z}$ is S^1. According to the previous thm. we get

$$\pi_1(S^1) \cong \mathbb{Z}.$$

Example B The group $\mathbb{Z}\oplus\mathbb{Z}$ acts of $\mathbb{R}^2$

$$(m,n)\cdot(x,y) = (x+m, y+n)$$

The action is properly discontinuous. $\mathbb{R}^2$ is simply connected, hence

$$\pi_1(\mathbb{R}^2/\mathbb{Z}\oplus\mathbb{Z}) \cong \mathbb{Z}\oplus\mathbb{Z}$$

The space

$$\mathbb{R}^2/\mathbb{Z}\oplus\mathbb{Z}$$

is the torus.

Example C For $n \geq 2$ the sphere S^n is simply connected. Every map $S^1 \to S^n$ is homotopic to a map $S^1 \to S^n$ which is not onto and such a map is homotopic to the constant map.

Now

$$\mathbb{R}P^n = S^n/\mathbb{Z}/2$$

where the action of $\mathbb{Z}/2$ on S^n is given by multiplying by (-1). $(\mathbb{Z}/2 = \{1,-1\})$

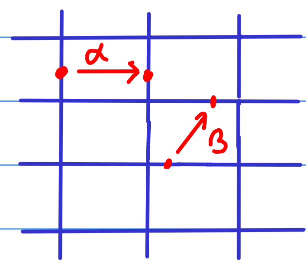
Hence $\pi_1(\mathbb{R}P^n) \cong \mathbb{Z}/2$

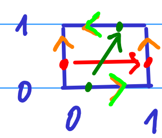
Example D The group G given by two generators α, β and the relation $\beta^{-1}\alpha\beta = \alpha^{-1}$ acts on $\mathbb{R}^2$

$$\alpha(x,y) = (x+1, y)$$

$$\beta(x,y) = (1-x, y+1)$$

The factor $\mathbb{R}^2/G$ is the Klein bottle.





Another method for computing the fundamental group

Free products of groups $\ast_\alpha G_\alpha$, $\alpha \in I$

$$g_1 g_2 \dots g_m \qquad 1 \neq g_i \in G_{\alpha_i}, \quad \alpha_i \neq \alpha_{i+1}$$

It is the limit of the diagram of groups G_α without any arrows and it has the following universal property

$$\begin{tikzcd}
 & G_\beta \arrow[dl, "i_\beta"'] \arrow[drr] & G_\gamma \arrow[dll, "i_\gamma"] \arrow[dr] & \\
\ast_\alpha G_\alpha \arrow[rrr, dashed] & & & G
\end{tikzcd}$$

Van Kampen Theorem enables us to compute the fundamental group of a union of spaces if we know the fundamental groups of single spaces and their intersections.

Assumptions: (1) $X = \bigcup U_\alpha$ where U_α are open in X and path connected, $\forall \alpha \in I$ $x_0 \in U_\alpha$.

(2) For every $\alpha, \beta \in I$, $U_\alpha \cap U_\beta$ is path connected

(3) For every $\alpha, \beta, \gamma \in I$, $U_\alpha \cap U_\beta \cap U_\gamma$ is path connected.

Define: $j_\alpha : U_\alpha \hookrightarrow X$ as inclusions, they induce

$$j_\alpha : \pi_1(U_\alpha) \longrightarrow \pi_1(X)$$

and these homomorphisms induce the homomorphism

$$\varphi : \mathop{*}_{\alpha} \pi_1(U_\alpha) \longrightarrow \pi_1(X).$$

Further, consider inclusions

$$i_{\alpha\beta} : U_\alpha \cap U_\beta \hookrightarrow U_\alpha$$

which induce homomorphisms

$$i_{\alpha\beta} : \pi_1(U_\alpha \cap U_\beta) \longrightarrow \pi_1(U_\alpha).$$

It holds that

$$j_\alpha \circ i_{\alpha\beta} = j_\beta \circ i_{\beta\alpha}$$

$$\begin{tikzcd}
 & U_\alpha \arrow[dr, hook, "i_\alpha"] & \\
U_\alpha \cap U_\beta \arrow[ur, "j_{\alpha\beta}"] \arrow[dr, hook, "j_{\beta\alpha}"'] & & X \\
 & U_\beta \arrow[ur, hook, "i_\beta"'] &
\end{tikzcd}$$

It is clear that $\ker \varphi$ must contain element

$$i_{\alpha\beta}(w)\; i_{\beta\alpha}(w^{-1})$$

for every $w \in \pi_1(U_\alpha \cap U_\beta)$. (It is equivalently $i_{\alpha\beta}(w) = i_{\beta\alpha}(w)$)

$$\begin{tikzcd}
 & \pi_1(U_\alpha) \arrow[d] \arrow[dr, "j_\alpha"] & \\
\pi_1(U_\alpha \cap U_\beta) \ni w \arrow[ur, "i_{\alpha\beta}"] \arrow[dr, "i_{\beta\alpha}"'] & *_\gamma \pi_1(U_\gamma) \arrow[r, "\varphi"] & \pi_1(X) \\
 & \pi_1(U_\beta) \arrow[u] \arrow[ur, "j_\beta"'] &
\end{tikzcd}$$

Van Kampen Theorem

Let $X = \bigcup_\alpha U_\alpha$, U_α open and path connected and let $U_\alpha \cap U_\beta$ be path connected for all α, β.

(1) Then

$$\varphi : *\, \pi_1 U_\gamma \longrightarrow \pi_1(X)$$

is an epimorphism.

(2) If $U_\alpha \cap U_\beta \cap U_\gamma$ are path connected for all α, β, γ, then $\ker \varphi$ is a normal subgroup N of $*_\gamma\, \pi_1(U_\gamma)$ generated by elements

$$\{\, i_{\alpha\beta}(w) \circ i_{\beta\alpha}(w^{-1}),\;\; w \in \pi_1(U_\alpha \cap U_\beta) \,\}$$

Hence

$$\pi_1(X, x_0) = \frac{*_\gamma \pi_1(U_\gamma)}{N}$$

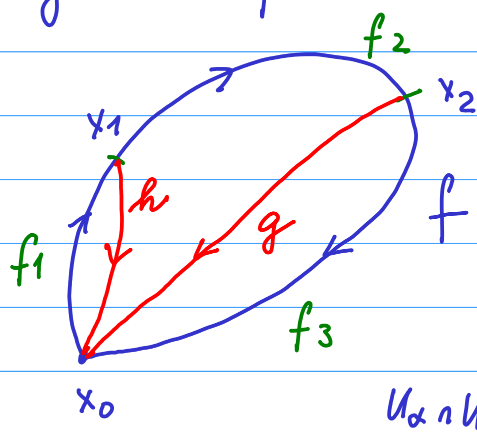
Example

$$\pi_1 (\vee X_\alpha) = *_\alpha \pi_1 (X_\alpha)$$

So

$\pi_1 (S^1 \vee S^1)$ is a free group with two generators.

Proof of surjectivity $\quad f : I \to X, \; f(0) = f(1) = x_0$



$\operatorname{im} f_1 \subseteq U_\alpha$
$\operatorname{im} f_2 \subseteq U_\beta$
$\operatorname{im} f_3 \subseteq U_\alpha$

$x_0, x_1, x_2 \in U_\alpha \cap U_\beta$

$U_\alpha \cap U_\beta$ is path connected

There are curves h, g $\operatorname{im} h, \operatorname{im} g \subseteq U_\alpha \cap U_\beta$. Hence

$$f \sim \underbrace{(f_1 \cdot h)}_{\pi_1(U_1)} \cdot \underbrace{(h^{-1} \cdot f_2 \cdot g)}_{\pi_2(U_2)} \cdot \underbrace{(g^{-1} f_3)}_{\pi_1(U_1)}$$

$\to$ • composition of curves from left to right

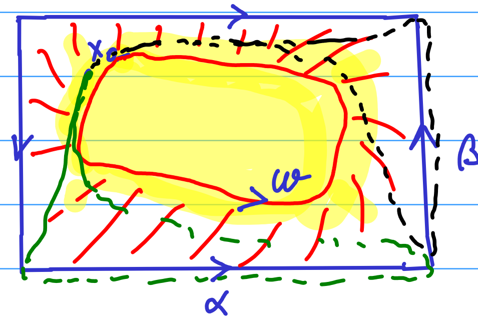
Consequently $[f_3 \circ g^{-1}] \cdot [g \circ f_2 \circ h^{-1}] \cdot [h \cdot f_1] \in \varphi^{-1}([f])$.
φ is onto.

The proof of the second statement is in the text Introduction to algebraic topology in section 11.

Corollary: Let $X = U \cup V$, U, V open and path connected, let V be moreover simply connected a $U \cap V$ be path connected. Then

$$\pi_1(X) \cong \frac{\pi_1(U)}{\operatorname{im} \pi_1(U \cap V)}$$

Example: Using corollary compute the fundamental group of the Klein bottle.



U red subset
V yellow subset
V is simply connected

U is homotopy equivalent to $S^1 \vee S^1$, hence

$$\pi_1(U) = \langle \alpha, \beta \rangle$$ free group on 2 generators

$U \cap V$ is homotopy equivalent to S^1 with generator w

$i_{VU}(w) = 1$ since V is simply connected

$i_{UV}(w) = \alpha \beta \alpha^{-1} \beta$

$\pi_1(K)$ is given by generator α, β and the relation $\alpha\beta\alpha^{-1}\beta = 1$

or $\alpha\beta\alpha^{-1} = \beta^{-1}$.

Remark (Fundamental group and the first homology group)

Regarding loops as 1-cycles we get a homomorphism

$$h : \pi_1(X, x_0) \longrightarrow H_1(X)$$

If X is path connected, then h is surjective and its kernel is the commutator subgroup of $\pi_1(X)$. (Hatcher, Thm 2A.1, pages 166-167).