Lecture 11 HOMOTOPY AND CW-COMPLEXES

Last time: notions of n-connectivity and n-equivalence

Compression lemma: (X,A) pair of CW-complexes, (Y,B) pair of spaces

$$\pi_n(Y,B,y_0)=0 \quad \text{for all } y_0\in B$$

whenever there is an n-cell in $X\setminus A$.

Then every $f:(X,A)\longrightarrow(Y,B)$ is homotopic rel A to a map $g: X\longrightarrow B$.

$$\begin{tikzcd}
A \arrow[r, "f/A"] \arrow[d, hook] & B \arrow[d, hook] \\
X \arrow[r, "f"'] \arrow[ur, "g"] \arrow[ur, phantom, "\sim"'] & Y
\end{tikzcd}$$

Theorem Let $h: Z\to Y$ be an n-equivalence. For every CW-complex X the induced map

$$h_*: [X,Z]\longrightarrow[X,Y]$$

is

(1) surjection if $\dim X\le n$,

(2) bijection if $\dim X\le n-1$.

Proof: If h is an inclusion we use compression lemma. If $\dim X\le n$ for this situation

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d, hook] & Z \arrow[d, hook, "h"] \\
X \arrow[r] \arrow[ur] & Y
\end{tikzcd}$$

It implies (1).

For $\dim X \leq n-1$, for this situation

$$\begin{tikzcd}
X\times\{0\} \cup X\times\{1\} \arrow[r, shift left, "f_0"] \arrow[r, shift right, "f_1"'] \arrow[d, hook] & Z \arrow[d, "h"] \\
X \times I \arrow[r, "f"'] \arrow[ur, red, "H"] & Y
\end{tikzcd}$$

which implies (2).

If h is not an inclusion, we replace Y by the cylinder of h:

$$\begin{tikzcd}
 & & Z \arrow[dl, "h"'] \arrow[d, "i_Z", "\sim"'] \arrow[drr, "h"] & & \\
X \arrow[urr, red] \arrow[r] & Y \arrow[r, "\simeq", "i_Y"'] & M_h \arrow[rr, "\simeq", "p"'] & & Y
\end{tikzcd}$$

Weak homotopy equivalence

A map $f: X \to Y$ is called a weak homotopy equivalence if

$$f_*\ \pi_n(X, x_0) \longrightarrow \pi_n(Y, f(x_0))$$

is an isomorphism for all $n \geq 0$ and $x_0 \in X$.

WHITEHEAD THEOREM

Let $h: Z \to Y$ be a weak equivalence between two CW-complexes. Then h is a homotopy equivalence.

Proof: If h is an inclusion we again apply the compression lemma:

$$\begin{tikzcd} Z \arrow[r, "id_Z"] \arrow[d, "h"'] & Z \arrow[d, "h"] \\ Y \arrow[r, "f"'] \arrow[ur, "g"] & Y \arrow[u, phantom, "\simeq"] \end{tikzcd}$$

$\pi_n(Y, Z) = 0$ for all n

g is a homotopy inverse of h.

If h is not an inclusion we use the mapping cylinder of h.

SIMPLICIAL APPROXIMATION LEMMA

Assumptions: $f : I^n \longrightarrow Z = W \cup e^k$ where W is a space and e^k is a k-cell.

Conclusion: There is a map $f_1 : I^n \longrightarrow Z$ and a simplex $\Delta^k \subset e^k$ such that

(1) $f_1 \simeq f$ rel $f^{-1}(W)$

(2) $f_1^{-1}(\Delta^k)$ is a union of finitely many convex polyhedra such that f_1 on the polyhedra is an affine projection (surjective) $\mathbb{R}^n \longrightarrow \mathbb{R}^k$. (If $k > n$, then $f_1^{-1}(\Delta^k)$ is empty.)

Hatcher Lemma 4.40 (350-351)

CELLULAR APPROXIMATION THEOREM

Let $f : X \to Y$ is a map between two CW-complexes, which is cellular on subcomplex $A \subset X$. Then there is a cellular map $g : X \to Y$ such that
$$g \sim f \text{ rel } A.$$

Corollary 1: $\pi_k(S^n) = 0$ for $k < n$.

Corollary 2: Let (X, A) be a pair of CW-complexes and let $X \setminus A$ contain cells of dim $> n$. Then the pair (X, A) is n-connected.

Proof of Cor 2 Every class in $\pi_k(X, A)$, $k \leq n$ contains a cellular representative $g\ I^k \to X^k \subseteq A$. Hence $[g] = 0$ in $\pi_k(X, A)$.

Proof of cellular appr. thm: By induction

$f_{-1} = f$

$f_n : X \to Y$, f_n is cellular on X^n

$f_n \sim f_{n-1}$ rel $X^{n-1} \cup A$

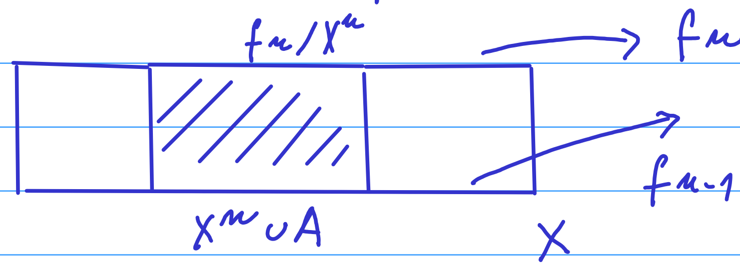
If we have such a sequence of maps we can define $g(x) = f_n(x)$ for $x \in X^n$ and we will have $g \sim f$ rel A.

Induction step: Suppose we have f_{n-1}. $f_{n-1}(e^n)$ does not lie in Y^n for an n-cell e^n. Then $f(e^n)$ has an intersection with a cell e^k in Y, $k > n$. According to simpl. appr. lemma, there is $h : \overline{e^n} \to Y$, $h \sim f_{n-1}/\overline{e}_n$ rel ∂e^n and there is a $\Delta^k \subset e^k$, $h(\overline{e}^n) \subset Y \setminus \Delta^k$.

∂e^k is a deformation retract of $\bar{e}^k \setminus \Delta^k$, hence there is $g : \bar{e}^n \to Y \setminus e^k$, $g \sim h$ rel ∂e^n. We repeat this procedure until we get a map $\bar{e}^n \to Y$ with image in Y^n, which is homotopic to $f_{n-1}/\bar{e}_n$ rel $\partial \bar{e}^n$.

So we get $f_n : X^n \to Y^n$ $\quad f_n \sim f_{n-1}$ rel $A \cup X^{n-1}$

Using HEP we extend f_n on the map $X \to Y$.



Approximation of spaces by CW-complexes

Let (X,A) be a pair, X a general space, A a CW-complex. A pair of CW-complexes (Z,A) is called

n-connected CW model for (X,A)

if there is a map $f : Z \to X$ such that

(1) $f/A = id_A$,

(2) $f_* \ \pi_i(Z) \to \pi_i(X)$ is an iso for $i > n$,

(3) $f_* \ \pi_n(Z) \to \pi_n(X)$ is a mono.

If A contains a point from every component of path connectivity of X, then 0-connected model $f : Z \to X$ is a weak homotopy equivalence.

CW approximation theorem

For every $n \in \mathbb{N}$ and every pair (X, A) where A is a CW-complex there is an n-connected CW-model
$$f : (Z, A) \longrightarrow (X, A)$$
such that $Z \setminus A$ has only cells of dim $> n$.

Proof by induction

$$A = Z_n \subset Z_{n+1} \subset \ldots\ldots Z_{k-1} \subset Z_k \subset \ldots\ldots \quad Z$$

Z_k arises from Z_{k-1} by attaching cells of dim k.

$$f : Z_k \to X, \quad f|A = id_A$$

$$f_* \quad \pi_i(Z_k) \to \pi_i(X) \text{ mono for } n \le i < k$$
$$\text{epi for } n < i \le k$$

Suppose X, A path connected, $x_0 \in A$ fixed.

We get Z_{k+1} from Z_k in two steps:

- We have $f : Z_k \to X$. Let $\varphi_\alpha : S^k \to Z_k$ which are generators of
  $$\ker(f_* \, \pi_k(Z_k) \to \pi_k(X))$$
  Put
  $$Y_{k+1} = Z_k \cup_{\varphi_\alpha} \bigcup D_\alpha^{k+1}$$
  $f : Z_k \to X$ can be extended to $f : Y_{k+1} \to X$ due to the fact that $f_*[\varphi_\alpha] = 0 \in \pi_k(X)$.

  $$\pi_i(Y_{k+1}) = \pi_i(Z_k) \quad \text{for } i \le k-1$$
  according to cellular appr. thm.

$$\begin{tikzcd} \pi_k(Z_k) \arrow[r] \arrow[rr, bend left, "epi"] & \pi_k(Y_{k+1}) \arrow[r, "epi"] & \pi_k(X) \end{tikzcd}$$

is an epi according to assumption on

$$f_* \; \pi_k(Z_k) \to \pi_k(X).$$

Hence extended f gives also epi

$$f_* : \pi_k(Y_{k+1}) \to \pi_k(X)$$

We prove that $f_* : \pi_k(Y_{k+1}) \to \pi_k(X)$ is a mono.
Let $[\varphi] \in \pi_k(Y_{k+1})$ and $f \circ \varphi \sim 0$.
$\varphi : S^k \to Y_{k+1}$ is homotopic to $\bar{\varphi} : S^k \to Y^k_{k+1} = Z_k$
and $[f\bar{\varphi}] = 0$ in $\pi_k(X)$.
That is why $[\bar{\varphi}] \in \ker f_*$ and consequently
$[\bar{\varphi}] = \sum [\varphi_\alpha]$ in $\pi_k(Z_k)$ but also in $\pi_k(Y_{k+1})$
Now $[\varphi_\alpha] = 0$ in $\pi_k(Y_{k+1})$, and so $[\bar{\varphi}] = 0$
in $\pi_k(Y_{k+1})$.

Conclusion $\quad f_* : \pi_i(Y_{k+1}) \to \pi_i(X)$ mono $n \le i \le k$
and epi $n < i \le k$.

- Let $\psi_\beta : S^{k+1}_\beta \to X$ be generators of $\pi_{k+1}(X)$.
Put

$$Z_{k+1} = Y_{k+1} \vee \bigvee_\beta S^{k+1}_\beta$$

and define $f = \psi_\beta$ on S^{k+1}_β.

Then $f_* : \pi_{k+1}(Z_{k+1}) \to \pi_{k+1}(X)$ is
an epimorphism.
Next $\pi_i(Y_{k+1}) \to \pi_i(Z_{k+1})$ iso for $i \le k-1$ and so
$$f_* : \pi_i(Z_{k+1}) \to \pi_i(X)$$
are iso according to assumptions of Y_{k+1}. Inclusion
$Y_{k+1} \hookrightarrow Z_{k+1}$ induces epi : $\pi_k(Y_{k+1}) \to \pi_k(Z_{k+1})$

Use cellular appr. thm. Every class in $\pi_k(Z_{k+1})$ can be represented by a map $S^k \to Z^k_{k+1}$, i.e by a map $S^k \longrightarrow Z_k \hookrightarrow Y_{k+1}$. Hence also

$$f_* : \pi_k(Z_{k+1}) \longrightarrow \pi_k(X) \text{ is an epi.}$$

It remains to show that

$$f_* : \pi_k(Z_{k+1}) \longrightarrow \pi_k(X)$$

is a mono.

We use the long exact sequence of the pair (Z_{k+1}, Y_{k+1})

$$\begin{tikzcd}
\pi_{k+1}(Z_{k+1}, Y_{k+1}) \arrow[r] & \pi_k(Y_{k+1}) \arrow[r, two heads, "epi"] \arrow[dr, "\cong", "f_*"', "iso"'] & \pi_k(Z_{k+1}) \arrow[d, "f_*"', "mono"] \\
& & \pi_k(X)
\end{tikzcd}$$

Corollary: n-connected pair of CW-complexes (X, A) is homology equivalent to a pair of CW-complexes (Z, A) where $Z - A$ has only cells in dim $\geq n+1$.

Proof: Let $(Z, A) \to (X, A)$ be an n-connected CW model. Then

$$f_* \, \pi_i(Z) \longrightarrow \pi_i(X) \quad \text{is an iso}$$

for all i. For $i \geq n+1$ by the properties of model, for $i \leq n-1$ by the fact that $\pi_i(A) \to \pi_i(X)$ is an isomorphis. For $i = n$: n-model gives $f_* \pi_n(Z) \to \pi_n(X)$ is a mono and n-connectivity gives $\pi_n(A) \to \pi_n(X)$ is an epi. Since $Z^n = A$, we get that

$$f_* \; \pi_n(Z) \longrightarrow \pi_n(X) \text{ is an iso.}$$