Lecture 12: Homotopy excision and Hurewicz Theorem

Homotopy excision (BLAKERS-MASSEY THEOREM)

Let A, B are subcomplexes in the CW-complex $X = A \cup B$. Let $C = A \cap B$ be connected. If the pair (A, C) is m-connected & the pair (B, C) is n-connected, then the inclusion

$$(A, C) \hookrightarrow (X, B)$$

is an $(n+m)$-equivalence.

Compare with excision theorem for homology groups! For the proof see the text to the lecture (Chapter 13) or Hatcher.

Corollary: Let (X, A) be an r-connected pair of CW-complexes & let A be s-connected. Then the map

$$X \to X/A$$

is an $(r+s+1)$-equivalence.

Proof: The pair (X, A) is r-connected according to the assumption and the pair (CA, A) is $(s+1)$-connected because

$$\pi_{i+1}(CA, A) \cong \pi_i(A)$$

from the long exact sequence of the pair (CA, A). The Blakers-Massey Theorem gives that

$$(X, A) \longrightarrow (X \cup CA, CA)$$

is $(r+s+1)$-equivalence. Further

$$\pi_i(X \cup CA, CA) \longleftarrow \pi_i(X \cup CA)$$

is an isomorphism (since CA is contractible) and

$$X \cup CA \longrightarrow X \cup CA/CA$$

is a homotopy equivalence (since CA is contractible in itself) and

$$X \cup CA/CA \longleftarrow X/A$$

is a homeomorphism

$$\begin{tikzcd}
(X,A) \arrow[r, hook, "(r+s+1)\text{-equiv}"] \arrow[ddrr, bend right, "(r+s+1)\text{-equiv}"'] & (X\cup CA,\, CA) \arrow[r, "\infty\text{-equiv}"] & X\cup CA/CA \arrow[dd, "\cong\quad \infty\text{-equiv}"] \\
& X\cup CA \arrow[u, hook, "\infty\text{-equiv}"] \arrow[ur, "\infty\text{-equiv}"'] & \\
& & X/A
\end{tikzcd}$$

Freudenthal Theorem

Let X be $(n-1)$-connected CW-complex, $n \geq 1$. Then the suspension homomorphism

$$S : \pi_i(X) \longrightarrow \pi_{i+1}(SX)$$
$$f \longmapsto Sf$$

is an isomorphism for $i \leq 2n-2$ and an epimorphism for $i = 2n-1$.

Proof: $SX = C_+X \cup C_-X$, $C_+X \cap C_-X = X$

The pairs (C_+X, X) an (C_-X, X) are n-connected

We get

$$\begin{tikzcd}
{[f]} \in \pi_i(X) \arrow[ddrr, "S"] & \pi_{i+1}(C_+X, X) \arrow[l, "\partial"', "\cong\ \text{LONG EXACT SEQ.}"] \arrow[r, "{[C_+f,\, f]}\ \cong \text{ for } i+1 \leq 2n-1,\ \text{epi for } i+1 = 2n"] & \pi_{i+1}(SX, C_-X) \ni {[Sf, C_-f]} \arrow[d, "\cong\ \text{previous statement}"] \\
& & \pi_{i+1}(SX/C_-X) \\
& & \pi_{i+1}(SX) \ni {[Sf]} \arrow[u, "\cong"']
\end{tikzcd}$$

We have to show that it is really suspension.

Stable homotopy groups

The Freudenthal Theorem holds not only for CW-complexes but also for all topological spaces. The proof is based on the fact that for every top. space Z we can find a CW-complex X and a weak homotopy equivalence $X \xrightarrow{f} Z$. Using the diagram

$$\begin{array}{ccc} \pi_i(X) & \xrightarrow{f} & \pi_i(Z) \\ S_X \downarrow & & \downarrow S_Z \\ \pi_{i+1}(SX) & \xrightarrow{Sf} & \pi_{i+1}(SZ) \end{array}$$

we get that S_Z is an iso if S_X is an iso and that S_Z is an epi if S_X is an epi.

If X is n-connected, then SX is $(n+1)$-connected. So in the sequence of suspension maps

$$\pi_i(X) \longrightarrow \pi_{i+1}(SX) \longrightarrow \pi_{i+2}(S^2X) \longrightarrow \dots \pi_{i+n}(S^nX) \to \dots$$

we get from a certain point isomorphisms.

If X is not connected, then SX is connected, S^2X is 1-connected, etc, S^nX is $(n-1)$-connected and so

$$\pi_i(S^nX) \longrightarrow \pi_{i+1}(S^{n+1}X)$$

is an iso for $i \leq 2n-2$. For such i

$$\pi_{i+j}(S^{n+j}X) \longrightarrow \pi_{i+j+1}(S^{n+j+1}X)$$

are iso for all $j \geq 0$, because

$$i+j \leq 2n+2j-2.$$

For a fixed i we take $n \geq i+2$, we get

$$i+n \leq 2n-2$$

and hence $\pi_{i+n}(S^nX) \longrightarrow \pi_{i+n+1}(S^{n+1}X)$

is an isomorphism.

We define <u>stable homotopy groups</u> as

$$\pi_i^s(X) = \operatorname*{colim}_{n\to\infty} \pi_{i+n}(S^n X).$$

<u>Theorem</u> The group $\pi_n(S^n)$ is isomorphic to $\mathbb{Z}$ with generator given by the identity
$id : S^n \to S^n$.
Moreover, the isomorphism
$$\pi_n(S^n) \xrightarrow[\cong]{} \mathbb{Z}$$
is given by the degree of maps.

<u>Proof:</u>

$$\begin{tikzcd}
\pi_1(S^1) \arrow[r, "epi"] \arrow[d, "deg"', "\cong"] & \pi_2(S^2) \arrow[r, "iso"] \arrow[d, "deg"] & \pi_3(S^3) \arrow[r] \arrow[d, "deg"] & \dots \\
\mathbb{Z} \arrow[r, "id"] & \mathbb{Z} \arrow[r, "id"] & \mathbb{Z} \arrow[r] & \dots
\end{tikzcd}$$

$id/S^1 \qquad id/S^2 = S(id/S^1)$

Since $\deg Sf = \deg f$, we get subsequently, that $\deg\ \pi_i(S^i) \xrightarrow[\deg]{} \mathbb{Z}$ are isomorphisms.

<u>Lemma</u>

$$\pi_n\Big(\prod_{\alpha\in A} X_\alpha\Big) \cong \prod_{\alpha\in A} \pi_n(X_\alpha)$$

<u>Lemma</u> For $n \geq 2$

$$\pi_n\Big(\bigvee_{\alpha\in A} S^n_\alpha\Big) = \bigoplus_{\alpha\in A} \mathbb{Z}$$

<u>Proof:</u> (1) If A is finite, then $\bigvee_A S^n_\alpha$ is a subcomplex in $\prod_{\alpha\in A} S^n_\alpha$. The pair $(\prod_A S^n_\alpha, \bigvee S^n_\alpha)$

is $(2n-1)$-connected since all the cells in $\prod S^n_\alpha \setminus \bigvee S^n_\alpha$ are of dimension $2n$ and higher. That is why for $n \geq 2$

$$\pi_n(\vee S^n_\alpha) \cong \pi_n(\prod S^n_\alpha) \cong \prod \pi_n(S^n_\alpha) \cong \oplus \pi_n(S^n_\alpha)$$

since the product of finite number of abelian groups is the same as their sum.

(2) A infinite. Then

$$\Phi : \oplus \pi_n(S^n_\alpha) \longrightarrow \pi_n(\vee S^n_\alpha)$$

is induced by maps $S^n_\alpha \longrightarrow \vee S^n_\alpha$. Φ is an isomorphism since every map into $\vee S^n_\alpha$ from S^n or $S^n \times I$ goes only into finite number of spheres.

Lemma Let $n \geq 2$ and let

$$X = \bigvee_{\alpha \in A} S^n_\alpha \cup_{\varphi_\beta} \cup e^{n+1}_\beta$$

where

$$\varphi_\beta : S^n_\beta \longrightarrow \vee S^n_\alpha$$

is an attaching map for e^{n+1}_β. Then

$$\pi_i(X) = \begin{cases} 0 & \text{for } i \leq n-1 \\ \bigoplus_{\alpha \in A} \pi_n(S^n_\alpha) / N \end{cases}$$

where N is a subgroup generated by $[\varphi_\beta]$.

Proof: The long exact sequence for the pair $(X, X^n = \bigvee_{\alpha \in A} S^n_\alpha)$ gives

$$\pi_{n+1}(X, X^n) \xrightarrow{\partial} \pi_n(X^n) \longrightarrow \pi_n(X) \longrightarrow \pi_n(X, X^n) = 0.$$

The pair (X, X^n) is n-connected, X^n is $(n-1)$-connected, hence

$$\pi_{n+1}(X, X^n) \xrightarrow[\cong]{} \pi_{n+1}(X/X^n) = \pi_{n+1}(\vee S^{n+1}_\beta) = \bigoplus_{\beta\in B} \mathbb{Z}$$

That is why

$$\pi_n(X) \cong \pi_n(X^n)/\operatorname{Im}\partial \cong \bigoplus_{\alpha\in A}\mathbb{Z} / N$$

We show that $\operatorname{Im}\partial \cong N$.

$$\begin{tikzcd} \pi_{n+1}(X, \vee_\alpha S^n_\alpha) \arrow[r, "\partial"] \arrow[d, "\cong"] & \pi_n(\vee S^n_\alpha) \\ \pi_{n+1}(X/\vee_\alpha S^n_\alpha) = \pi_{n+1}(\vee_\beta S^{n+1}_\beta) & \end{tikzcd}$$

Generators in the group $\pi_{n+1}(X, \vee_\alpha S^n_\alpha)$ are maps

$$\begin{tikzcd} D^{n+1}_\beta \arrow[r, "\Phi_\beta"] & X \\ \partial D^{n+1}_\beta \arrow[u, hook] \arrow[r, "\varphi_\beta"'] & \vee S^n_\alpha \arrow[u, hook] \end{tikzcd}$$

corresponds to maps $D^{n+1}_\beta \to \vee_\beta S^{n+1}_\beta$, $\partial D^{n+1}_\beta \to *$ which are generators in $\pi_{n+1}(V_\beta S^{n+1}_\beta)$

and $\partial[\Phi_\beta] = [\varphi_\beta]$. Hence $\operatorname{Im}\partial = N$ = the group generated by $[\varphi_\beta]$.

Hurewicz homomorphism

For every space X we define a map

$$h : \pi_n(X) \longrightarrow H_n(X)$$

as

$$h[f] = f_*(s)$$

where $f : S^n \to X$ and $s \in H_n(S^n) \cong \mathbb{Z}$ is a generator, $f_* \, H_n(S^n) \to H_n(X)$, $f_*(s) \in H_n(X)$.

Similarly

$$h : \pi_n(X, A) \longrightarrow H_n(X, A)$$

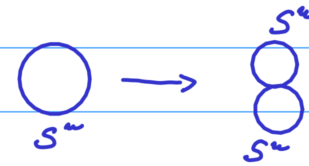
$$h[f] = f_*(s)$$

where $\quad f : (D^n, S^{n-1}) \longrightarrow (X, A)$
and $\quad s \in H_n(D^n, S^{n-1}) \cong \mathbb{Z}$
is a generator.

Lemma: $\quad h : \pi_n(X) \longrightarrow H_n(X)$
is a group homomorphism.

Proof: Consider

$$c : S^n \longrightarrow S^n \vee S^n$$



$$\begin{tikzcd}
H_n(S^n) \arrow[r, "c_*"] & H_n(S^n \vee S^n) \arrow[r, "(f \vee g)_*"] \arrow[d, shift left, "q_{1*} \oplus q_{2*}"] & H_n(X) \\
 & H_n(S^n) \oplus H_n(S^n) \arrow[u, shift left, "i_{1*} + i_{2*}"] &
\end{tikzcd}$$

$$h([f] + [g]) = (f+g)_* (s) = (f \vee g)_* \, c_* (s)$$

$$= (f \vee g)_* (i_{1*} + i_{2*}) (q_{1*} \oplus q_{2*}) \, c_* (s) =$$

$$= (f \vee g)_* (i_{1*} + i_{2*}) (s \oplus s) = f_*(s) + g_*(s) .$$

Homomorfismus $h : \pi_n(X) \longrightarrow H_n(X)$, resp.
$h : \pi_n(X, A) \longrightarrow H_n(X, A)$, is called
HUREWICZ HOMOMORPHISM.

Hurewicz homomorphism is natural: For
$f : (X, A) \longrightarrow (Y, B)$ we have commutative
diagram

$$\begin{tikzcd}
\pi_n(X, A) \arrow[r, "h"] \arrow[d, "f_*"'] & H_n(X, A) \arrow[d, "f_*"] \\
\pi_n(Y, B) \arrow[r, "h"] & H_n(Y, B)
\end{tikzcd}$$

and moreover

$$\begin{tikzcd}
\pi_{n+1}(X,A) \arrow[r, "h"] \arrow[d, "\partial"] & H_{n+1}(X,A) \arrow[d, "\partial"] \\
\pi_n(A) \arrow[r, "h"] & H_n(A)
\end{tikzcd}$$

We will prove it in the tutorial.

HUREWICZ THEOREM

Let $n \geq 2$. If X is $(n-1)$-connected, then $\tilde{H}_i(X) = 0$ for $i \leq n-1$ and

$$h : \pi_n(X) \longrightarrow H_n(X)$$

is an isomorphism.

Proof: Let X be a CW-complex and

$$X = \bigvee_{\alpha \in A} S^n_\alpha \cup_{\varphi_\beta} \bigcup e^{n+1}_\beta .$$

Then

$$\tilde{H}_i(X) = 0 \quad \text{for } i \leq n-1 .$$

Further

$$\begin{tikzcd}
\pi_{n+1}(X, X^n) \arrow[r, "\partial"] \arrow[d, "\cong\, h"'] & \pi_n(X^n) \arrow[r] \arrow[d, "\cong\, h"'] & \pi_n(X) \arrow[r] \arrow[d, "h"] & 0 \\
H_{n+1}(X, X^n) \arrow[r, "\partial"] & H_n(X^n) \arrow[r] & H_n(X) \arrow[r] & 0
\end{tikzcd} \qquad (*)$$

($h: \pi_n(S^n) \to H_n(S^n)$ and $h: \pi_{n+1}(S^{n+1}) \to H_{n+1}(S^{n+1})$ are iso.)

$$\begin{tikzcd}
\pi_{n+1}(X, X^n) \arrow[r, phantom, "\cong"] \arrow[d, "h"] & \pi_{n+1}(X/X^n) \arrow[r, phantom, "\cong"] \arrow[d, "h"] & \bigoplus \pi_n(S^{n+1}_\beta) \arrow[d, "h"] \\
H_{n+1}(X, X^n) \arrow[r, phantom, "\cong"] & H_{n+1}(X/X^n) \arrow[r, phantom, "\cong"] & \bigoplus H_n(S^{n+1}_\beta)
\end{tikzcd}$$

is an iso

Since in the diagram $(*)$ the first and second homomorphisms are iso, the third one is also iso.

For general X we know that $\pi_n(X) = \pi_n(X^{n+1})$

and $H_n(X) = H_n(X^{n+1})$. So the proof is completed for any CW-complex X.

<u>Remark:</u> $h \; \pi_n(S^n) \longrightarrow H_n(S^n)$ is an iso, because the generator of $\pi_n(S^n)$ is $[id : S^n \to S^n]$ and $h[id_{S^n}] = id_*(s) = s$ is a generator in $H_n(S^n)$.