Lecture 13 : Hurewicz Theorem and Whitehead Thm.

Last time : Hurewicz Theorem : If X is $(n-1)$-connected, then $\tilde{H}_i(X) = 0$ for $i \leq n-1$ and the Hurewicz homomorphism

$$h : \pi_n(X) \longrightarrow H_n(X)$$

is an isomorphism for $n \geq 2$.

Version for $n = 1$. If X is path connected then Hurewicz homomorphism

$$h : \pi_1(X) \longrightarrow H_1(X)$$

has the kernel which is the commutator of $\pi_1(X)$.

Relative version of Hurewicz Theorem

Let $n \geq 2$. Let a pair (X, A) be $(n-1)$-connected and A be 1-connected. Then $H_i(X, A) = 0$ for $i \leq n-1$ and the Hurewicz homomorphism

$$h : \pi_n(X, A) \longrightarrow H_n(X, A)$$

is an isomorphism.

Proof: If (X, A) is a pair of CW-complexes, then from $(n-1)$-connectedness of (X, A) and 1-connectedness of A follows that $(X, A) \longrightarrow X/A$ is an $(n+1)$-equivalence, it follows that $\pi_i(X, A) \cong \pi_i(X/A)$ for $i \leq n$.

Moreover, $H_i(X, A) \cong H_i(X/A)$. So we can apply Hurewicz Thm on X/A :

$$\begin{array}{ccc} \pi_n(X, A) & \xrightarrow{\cong} & \pi_n(X/A) \\ \cong \downarrow h & & \cong \downarrow h \\ H_n(X, A) & \xrightarrow{\cong} & H_n(X/A) \end{array}$$

to get that $h : \pi_n(X,A) \longrightarrow H_n(X,A)$ is an iso.

Homological version of Whitehead Theorem

We have already proved :
If $f : X \to Y$ is a map between CW-complexes, which is an isomorphism $f_* \ \pi_i(X, x_0) \longrightarrow \pi_i(Y, f(x_0))$ on the level of all homotopy groups, then f is a homotopy equivalence.

Using Hurewicz Theorem we are able to prove homological version of this theorem :

WHITEHEAD THEOREM (homological version)

Let X and Y be simply connected CW-complexes. Let $f : X \longrightarrow Y$ be a map which induces isomorphisms $f_* : H_i(X) \longrightarrow H_i(Y)$ on all homology groups. Then X and Y are homotopy equivalent via homotopy equivalence f.

Proof : We will prove that $f_* : \pi_i(X, x_0) \longrightarrow \pi_i(Y, y_0)$ are isomorphisms and apply the previous version of Whitehead Thm.

Suppose that $f : X \hookrightarrow Y$ is an iclusion. If not we apply this on mapping cylinder

$$X \hookrightarrow M_f .$$

We prove by induction that $f_* \ \pi_i(X) \longrightarrow \pi_i(Y)$ is an iso.

Obviously : $f_* : \pi_1(X) \cong 0 \longrightarrow \pi_1(Y) \cong 0$ is an iso.
and $\pi_1(Y, X) \cong 0$. Next

$$\begin{array}{ccccccc} \pi_2(X) & \longrightarrow & \pi_2(Y) & \longrightarrow & \pi_2(Y,X) & \longrightarrow & \pi_1(X)=0 \\ h\downarrow & & \downarrow h & & \cong\downarrow h & & \\ H_2(X) & \xrightarrow[\cong]{} & H_2(Y) & \longrightarrow & H_2(Y,X) & \longrightarrow & H_1(X)=0 \end{array}$$

Hence $H_2(Y,X) \cong 0$ and $\pi_2(Y,X) \cong 0$.
Further $H_3(Y,X) \cong 0 \Rightarrow \pi_3(Y,X) \cong 0$ etc.
It then implies $f_*\ \pi_2(Y) \longrightarrow \pi_2(Y)$ is an iso,
$f_*\ \pi_i(X) \longrightarrow \pi_i(Y)$ is an iso. ∎