Exercise 1. *There is a lemma, that says: Given the following diagram, where rows are long exact sequences and m is an iso*

$$\begin{tikzcd}
K_n \arrow[r, "i"] \arrow[d, "f"'] & L_n \arrow[r, "j"] \arrow[d, "g"'] & M_n \arrow[r, "h"] \arrow[d, "m"'] & K_{n-1} \arrow[r] \arrow[d] & L_{n-1} \arrow[r] \arrow[d] & M_{n-1} \arrow[d] \\
\overline{K}_n \arrow[r, "\bar{i}"] & \overline{L}_n \arrow[r, "\bar{j}"] & \overline{M}_n \arrow[r] & \overline{K}_{n-1} \arrow[r] & \overline{L}_{n-1} \arrow[r] & \overline{M}_{n-1}
\end{tikzcd}$$

we get a long exact sequence

$$K_n \xrightarrow{(i,f)} L_n \oplus \overline{K}_n \xrightarrow{g-\bar{i}} \overline{L}_n \xrightarrow{i \circ m^{-1} \circ \bar{j}} K_{n-1} \longrightarrow \cdots$$

We can denote $\partial = h \circ m^{-1} \circ \bar{j}$.
Show exactness in $L_n \oplus \bar{K}_n$ and also in $\bar{L}_n$.

Exercise 2. *There is a long exact sequence of the triple* (X, A, B)*, i.e.* $(B \subseteq A \subseteq X)$*:*

$$\cdots \to H_n(A, B) \xrightarrow{i} H_n(X, B) \xrightarrow{j_X} H_n(X, A) \xrightarrow{D_*} H_{n-1}(A, B) \to \cdots ,$$

with $H_n(X, A) \xrightarrow{\partial_*} H_{n-1}(A) \xrightarrow{j_A} H_{n-1}(A, B)$*. We get this sequence from a special short exact sequence of chain complexes. Show that it is exact and that the triangle commutes, that is* $D_* = j_A \circ \partial_*$*.*

Exercise 3. *Apply previous exercise to the triple* $(D^k, S^{k-1}, *)$*, where* $*$ *is a point.*

Exercise 4. *Show that the chain in* $C_k(\Delta^k, \partial\Delta^k)$ *given by* $\mathrm{id}\colon \Delta^k \to \Delta^k$ *is the representative of the generator of*

$$H_k(\Delta^k, \partial\Delta^k) \cong \mathbb{Z}.$$

(Use induction and the long exact sequence for triple.)

Exercise 5. *Using the Mayor-Vietoris exact sequence compute the homology groups of the torus.*

Exercise 6. *Prove Snake Lemma.*

$$\begin{tikzcd}
 & \ker f \arrow[r] & \ker g \arrow[r] & \ker h \arrow[ddll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-1ex]\tikztotarget.north) \tikztonodes -- (\tikztotarget)}, "\partial"'] & & \\
0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "f"] & B \arrow[r, "j"] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 & \text{exact} \\
0 \arrow[r] & \bar{A} \arrow[r, "\bar{i}"'] & \bar{B} \arrow[r, "\bar{j}"'] & \bar{C} \arrow[r] & 0 & \text{exact} \\
 & \operatorname{coker} f \arrow[r] & \operatorname{coker} g \arrow[r] & \operatorname{coker} h & &
\end{tikzcd}$$

$$\operatorname{coker} f = \bar{A}/\operatorname{im} f$$

Red sequence is exact. $c \in \ker h$

$\partial(c) = [\bar{a}]$ where $\bar{i}(\bar{a}) = g(b)$, $j(b) = c$.

Exercise 7. *Prove 5-lemma.*

If

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\cong"'] & B \arrow[r] \arrow[d, "\cong"] & C \arrow[r] \arrow[d, "g"] & D \arrow[r] \arrow[d, "\cong"] & E \arrow[d, "\cong"] & \leftarrow \text{exact} \\
\overline{A} \arrow[r] & \overline{B} \arrow[r] & \overline{C} \arrow[r] & \overline{D} \arrow[r] & \overline{E} & \leftarrow \text{exact}
\end{tikzcd}$$

then g is an isomorphism.

Proof using Snake Lemma.