Exercise 1. *Use $\mathbb{Z}/2$ coefficients to show, that every cts map $f\colon S^n \to S^n$ satisfying $f(-x) = -f(x)$ has an odd degree.*

Solution:

$$g : \mathbb{R}P^n \to \mathbb{R}P^n \qquad \mathbb{R}P^n \cong S^n/_{x\simeq -x}$$

$$g([x]) = [f(x)] \qquad p: S^n \to \mathbb{R}P^n \quad x \mapsto [x]$$

We use $H_*(\mathbb{R}P^n)$ and $H_*(S^n)$ with the $\mathbb{Z}/2$-coeff.

$$0 \to C_*(\mathbb{R}P^n; \mathbb{Z}/2) \to C_*(S^n; \mathbb{Z}/2) \xrightarrow{p_*} C_*(\mathbb{R}P^n; \mathbb{Z}/2) \to 0$$

$$\sigma_1+\sigma_2 \longmapsto \Sigma\, 2\sigma = 0$$

$$\begin{array}{ccc} & & S^n \\ & \sigma_1, \sigma_2 \nearrow & \downarrow p \\ \sigma: & \Delta^i \xrightarrow{\sigma} & \mathbb{R}P^n \end{array}$$

$$\sigma \longmapsto \sigma_1+\sigma_2 \longmapsto 2\sigma = 0$$

Short exact sequence. Induces LES of hom. groups

$$\begin{array}{ccccccc}
H_i(\mathbb{R}P^n;\mathbb{Z}/2) & \to & H_i(S^n;\mathbb{Z}/2) & \xrightarrow{0} & H_i(\mathbb{R}P^n;\mathbb{Z}/2) & \xrightarrow[\cong]{\delta_*} & H_{i-1}(\mathbb{R}P^n;\mathbb{Z}/2) \\
\downarrow g_i & & \downarrow f_i \ {=0} & & \mathrm{id}\downarrow g_i \ \mathbb{Z}/2 & & \mathrm{id}\downarrow g_{i-1} \\
H_i(\mathbb{R}P^n;\mathbb{Z}/2) & \to & H_i(S^n;\mathbb{Z}/2) & \to & H_i(\mathbb{R}P^n;\mathbb{Z}/2) & \xrightarrow{\cong} & H_{i-1}(\mathbb{R}P^n;\mathbb{Z}/2)
\end{array}$$

$f_* : H_n(S^n;\mathbb{Z}) \to H_n(S^n;\mathbb{Z})$ has odd degree

$f_* : \underset{\mathbb{Z}/2}{H_n(S^n;\mathbb{Z}/2)} \to \underset{\mathbb{Z}/2}{H_n(S^n;\mathbb{Z}/2)}$ is an identity

First by induction we prove that

$$g_i : \underset{\mathbb{Z}/2}{H_i(\mathbb{R}P^n;\mathbb{Z}/2)} \to \underset{\mathbb{Z}/2}{H_i(\mathbb{R}P^n;\mathbb{Z}/2)} \text{ is an identity}$$

$g_0 : H_0(\mathbb{R}P^n;\mathbb{Z}/2) \to H_0(\mathbb{R}P^n;\mathbb{Z}/2)$ identity

g_{i-1} is identity implies g_i is also identity for $i < n$

$i = n$ homology with $\mathbb{Z}/2$ coeff.

$$\begin{tikzcd}
 & \mathbb{RP}^n_{n+1} & \mathbb{RP}^n_{n} & S^n_{n} & \mathbb{RP}^n_{n} & \mathbb{RP}^n_{n-1} \\
0 \arrow[r] \arrow[d, "g_{n+1}"'] & \mathbb{Z}/2 \arrow[r, "inj", "\cong"'] \arrow[d, "id"', "g_n"] & \mathbb{Z}/2 \arrow[r, "0"] \arrow[d, "id"', "f_n"] & \mathbb{Z}/2 \arrow[r, "\cong"] \arrow[d, "g_n"] & \mathbb{Z}/2 \arrow[r, "0"] \arrow[d, "g_{n-1}"] & {} \\
0 \arrow[r] & \mathbb{Z}/2 \arrow[r, "inj", "\cong"'] & \mathbb{Z}/2 \arrow[r, "0"] & \mathbb{Z}/2 \arrow[r, "\cong"] & \mathbb{Z}/2 \arrow[r, "0"] & {}
\end{tikzcd}$$

$\Rightarrow$ id

It implies that g_n is also identity.
So f_n is also identity.
So we have proved that $\deg f$ is odd.

Exercise 2. *Let* $\varphi \in C^k(X;R), \psi \in C^l(X;R)$. *Prove* $\delta(\varphi \cup \psi) = \delta\varphi \cup \psi + (-1)^k \varphi \cup \delta\psi$. *Use* $\tau = [e_0, \dots, e_{k+l+1}] \in C_{k+l+1}(X)$.

$\varphi \cup \psi \in C^{k+l}(X;R) \qquad \delta(\varphi\cup\psi) \in C^{k+l+1}(X;R)$

$\tau : \Delta^{k+l+1} \to X$, $\quad \Delta^{k+l+1} = [e_0, \dots \quad e_{k+l+1}]$

$\delta(\varphi\cup\psi)(\tau) = (\varphi\cup\psi)(\partial\tau) =$

$= (\varphi\cup\psi)\left(\sum_{i=0}^{k+l+1} (-1)^i \tau/[e_0 \dots \hat{e}_i \dots e_{k+l+1}]\right)$

$= \sum_{i=0}^{k} (-1)^i \underbrace{\varphi(\tau/[e_0 \dots \hat{e}_i \dots e_{k+1}])}_{\in R} \cdot \psi(\tau[e_{k+1}, \dots e_{k+l+1}])$

$+ \sum_{i=k+1}^{k+l+1} (-1)^i \varphi(\tau/[e_0, \dots e_k]) \cdot \psi(\tau[e_k, \dots \hat{e}_i, \dots e_{k+l+1}])$

Left hand side

Right hand side

$(\delta\varphi \cup \psi)(\tau) = \delta\varphi(\tau/[e_0, \dots e_{k+1}]) \cdot \psi(\tau/[e_{k+1}, \dots e_{k+l+1}])$

$= \sum_{i=0}^{k+1} (-1)^i \varphi(\tau/[e_0 \dots \hat{e}_i \dots e_{k+1}]) \cdot \psi(\tau/[e_{k+1} \dots e_{k+l+1}])$ $\quad k+1$

The second term of the right hand side

$(-1)^k(\varphi \cup \delta\psi)(\tau) = (-1)^k \varphi(\tau/[e_0 \dots e_k]) \cdot \psi(\partial\tau/[e_k, \dots e_{k+l+1}])$ $\quad j=0$

$= (-1)^k \sum_{j=0}^{l+1} (-1)^j \varphi(\tau/[e_0 \dots e_k]) \cdot \psi(\tau/[e_k, \dots \hat{e}_{k+j}, \dots e_{k+l+1}])$

RHS = LHS + (terms) A − A

RHS = LHS

Exercise 3. *Compute the structure of graded algebra $H^*(S^n \times S^n;\mathbb{Z})$ for n even and n odd. Use the following:*

If $H^n(Y;R)$ is free finitely generated group for all n and $(X,A), Y$ are CW-complexes, then

$$\times\colon H^*(X,A;R)\otimes H^*(Y;R)\to H^*(X\times Y, A\times Y;R)$$

is an isomorphism of graded rings.

$H^*(Y;R)$ is a free graded R-module

$\underline{H_0} \oplus \underline{H_1} \oplus \underline{H_2} \oplus \dots \oplus \underline{H_n} \oplus \dots$

$\downarrow$ it has also a ring structure

$\cup \quad H^k \otimes H^l \longrightarrow H^{k+l}$

$\cup: H^* \otimes H^* \longrightarrow H^*$

$\underline{H^*(X;R)} \otimes \underline{H^*(Y;R)}$ has also a ring structure

$$(a\otimes b)\cdot(c\otimes d) \underset{def}{=} (-1)^{|b||c|}(a\cup c)\otimes(b\cup d)$$

$$H^*(X;R)\otimes H^*(Y;R) \xrightarrow{\mu} H(X\times Y;R)$$

$$\mu(a\otimes b) = p_X^*(a)\cup p_Y^*(b)$$

$p_X^*(a)\in H^*(X\times Y)$ $\quad$ $p_Y^*(b)\in H^*(X\times Y)$

$$\begin{array}{ccccc} & & X\times Y & & \\ & \swarrow^{p_X} & & \searrow^{p_Y} & \\ a\in H^*(X) & X & & Y & H^*(Y)\ni b\end{array}$$

1) μ is a ring homo

2) Thm: If $H^*(Y)$ free fin. gen. R-modul then μ is an iso of graded rings

one copy $\downarrow$ $\quad$ $\downarrow$ second copy

$$H^*(S^n;\mathbb{Z}) \otimes H^*(S^n;\mathbb{Z}) \longrightarrow H^*(S^n \times S^n;\mathbb{Z})$$

Describe $H^*(S^n \times S^n;\mathbb{Z})$ as a graded ring.

$H^*(S^n;\mathbb{Z})$

$1 \in H^0(S^n;\mathbb{Z})$

$a \in H^n(S^n;\mathbb{Z}) \cong \mathbb{Z}$ $\quad \bullet \longrightarrow 1 \in \mathbb{Z}$

$H^i(S^n) \cong 0 \quad i \neq 0, n$

$1 \cup a = a = a \cup 1$

$a \cup a \in H^{2n}(S^n;\mathbb{Z}) = 0 \qquad a^2 = 0$

first copy $H^*(S^n;\mathbb{Z}) \cong \mathbb{Z}[a]/(a^2) \qquad a \in H^n$

second $H^*(S^n;\mathbb{Z}) \cong \mathbb{Z}[b]/(b^2) \qquad b \in H^n$

Describe $H^*(S^n;\mathbb{Z}) \otimes H^*(S^n;\mathbb{Z})$

generators
$$\begin{aligned}
1 \otimes 1 &\rightsquigarrow 1 \\
a \otimes 1 &\rightsquigarrow \alpha \\
1 \otimes b &\rightsquigarrow \beta \\
a \otimes b &= (a \otimes 1) \cdot (1 \otimes b) = \alpha \cdot \beta
\end{aligned}$$

$0 = a^2 \qquad b^2 = 0$

$$\beta \cdot \alpha = (1 \otimes b) \cdot (a \otimes 1) = (-1)^{n \cdot n} (1 \cup a) \otimes (b \cup 1)$$
$$= (-1)^n \, a \otimes b = (-1)^n \alpha \cdot \beta$$

$$\beta \cdot \alpha - (-1)^n \alpha \cdot \beta = 0$$

$$H^*(S^n \times S^n, \mathbb{Z}) \cong H^*(S^n;\mathbb{Z}) \otimes H^*(S^n;\mathbb{Z})$$
$$\cong \mathbb{Z}[\alpha, \beta] \Big/ \left(\alpha^2, \beta^2, \beta \cdot \alpha - (-1)^n \alpha \cdot \beta\right)$$

n is even $\beta\alpha - \alpha\beta = 0$ $\beta\alpha = \alpha\beta$

n odd $\beta\alpha + \alpha\beta = 0$ $\beta\alpha = -\alpha\beta$

Exercise 4. *Prove that there is no multiplication on even dimensional spheres. Multiplication on the sphere S^n is a map $m\colon S^n \times S^n \to S^n$ such that there is an element $1 \in S^n$ satisfying $m(x,1) = x, m(1,x) = x$.*

Hint: compute $m^*\colon H^*(S^n) \to H^*(S^n \times S^n)$, describe both rings.

There is no multiplication on even spheres.

$$m : S^n \times S^n \longrightarrow S^n$$

such that there is an element 1 in S^n such that

$$m(x,1) = x$$
$$m(1,x) = x$$

multiplicatie with a unit

- $S^1 \subseteq \mathbb{C}$ multiplication $\cdot$ $\mathbb{C} \times \mathbb{C} \to \mathbb{C}$ $\cdot$ induces $m : S^1 \times S^1 \longrightarrow S^1$
- $S^3 \subset \mathbb{H}$ multiplication $\cdot$ $\mathbb{H} \times \mathbb{H} \longrightarrow \mathbb{H}$ $m : S^3 \times S^3 \longrightarrow S^3$
- $S^7 \subset \mathbb{O}$ $\cdot$ $\mathbb{O} \times \mathbb{O} \longrightarrow \mathbb{O}$ m $S^7 \times S^7 \longrightarrow S^7$

S^n n even Suppose we have a multiplication $m : S^n \times S^n \longrightarrow S^n$

$$m^* : H^*(S^n; \mathbb{Z}) \longrightarrow H^*(S^n \times S^n; \mathbb{Z})$$

$$\overset{\shortparallel}{\mathbb{Z}[x]/(x^2)} \longrightarrow \mathbb{Z}[\alpha, \beta]/(\alpha^2 = 0, \beta^2 = 0, \alpha\beta = \beta\alpha)$$

$$m^*(x) \in H^n(S^n \times S^n; \mathbb{Z}) = k\alpha + l\beta \quad k, l \in \mathbb{Z}$$

We prove that $k = l = 1$. $\qquad m^*(x) = \alpha + \beta$

Suppose it is true

$$\underline{0 =} \; m^*(0) = m^*(x^2) = \left[m^*(x)\right]^2 = (\alpha + \beta) \cup (\alpha + \beta) =$$

$$= \underbrace{\alpha \cup \alpha}_{0} + \alpha \cup \beta + \beta \cup \alpha + \underbrace{\beta \cup \beta}_{0} \;\; \underline{= 2 \alpha \cup \beta \neq 0}$$

contradiction

It remains to prove that $m^*(x) = \alpha + \beta$.

$m^*(x) = k \alpha + l \beta$

$$\begin{tikzcd}
x \arrow[r, mapsto] & (x, 1) \\
S^n \arrow[r, "i_1"] \arrow[dr, "id"'] & S^n \times S^n \arrow[d, "p_1"] \\
 & S^n
\end{tikzcd}$$

$p_1^*(x) = \alpha \qquad i_1^*(\alpha) = x$

$i_2^*(\beta) = x$

$$\begin{tikzcd}
x \arrow[r, mapsto] & (x, 1) \\
S^n \arrow[r, "i_1"] \arrow[dr, "id"'] & S^n \times S^n \arrow[d, "m"] \\
 & S^n
\end{tikzcd}$$

$$\underbrace{i_1^* m^*(x)}_{x} = i_1^*(k\alpha + l\beta) = kx \Rightarrow k = 1$$

similarly $\underline{l = 1}$

$m^*(x) = \alpha + \beta$.

Exercise 5. *Show commutativity of the diagram below. Use five lemma to prove that taking any two μ's iso's, the third μ is iso as well.*

free R-module

diagram

$$\begin{tikzcd}
H^*(X,A)\otimes H^*(Y) \arrow[rr] \arrow[dd, "\mu"', "\cong"] & & H^*(X)\otimes H^*(Y) \arrow[dl] \arrow[dd, "\mu", "\cong"'] \\
& H^*(A)\otimes H^*(Y) \arrow[ul, "\delta^*\otimes\mathrm{id}"'] \arrow[dd, "\mu", "\cong"'] & \\
H^*(X\times Y, A\times Y) \arrow[rr] & & H^*(X\times Y) \arrow[dl] \\
& H^*(A\times Y) \arrow[ul, "\delta^*"'] &
\end{tikzcd}$$

n+1

connecting homo

n

LES for (X, A) tensored by $H^*(Y)$

Preparation for the proof that

- $H^*(X,A)\otimes H^*(Y) \xrightarrow{\mu} H^*(X\times Y, A\times Y)$

is an isomorphism (under some assum

on Y.

Back square commutes – naturality of cross product

Right —"— —"— – —"— μ (×)

Commutativity of left square

$$\begin{tikzcd}
H^*(X,A)\otimes H^*(Y) \arrow[d, "\mu"'] & H^*(A)\otimes H^*(Y) \arrow[l, "\delta^*\otimes \mathrm{id}_Y"'] \arrow[d, "\mu"] \\
H^*(X\times Y, A\times Y) & H^*(A\times Y) \arrow[l, "\delta^*"']
\end{tikzcd}$$

$\alpha\in C^k(A)$ $\quad \beta\in C^\ell(Y)$ $\quad \alpha = \text{map}(C_k(A)\to R)$

$\delta\bar\alpha\otimes\beta \longleftarrow \alpha$ $\qquad$ We extend α on $C_k(X)$ ..... $\bar\alpha$

$\downarrow$ $\qquad$ We take $\delta\bar\alpha\in C^{k+1}(X,A)$

$p_1^*(\delta\bar\alpha)\cup p_2(\beta)$ $\qquad$ and determines $[\delta\bar\alpha]\in H^{k+1}(X,A)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \\ \delta^*[\alpha]$

$$\begin{tikzcd} \alpha \quad \beta \arrow[d, "w"] \\ p_1^*(\alpha) \cup p_2^*(\beta) \arrow[l, "\delta^*"'] \end{tikzcd}$$

$\bar{\alpha}$ extension of α from A to X

we extend $p_1^*(\alpha) \cup p_2^*(\beta)$ from $A \times Y$ on $X \times Y$ in such a way that we take $p_1^*(\bar{\alpha}) \cup p_2^*(\beta)$

$$\delta^*(p_1(\bar{\alpha}) \cup p_2^*(\beta)) = \delta(p_1(\bar{\alpha}) \cup p_2^*(\beta)) =$$

$$= \delta p_1^*(\bar{\alpha}) \cup p_2^*(\beta) + (-1)^k p_1^*(\bar{\alpha}) \cup \underbrace{\delta p_2^*(\beta)}_{\underbrace{p_2^* \delta\beta}_{0}}$$

$\delta\beta = 0$

$$= p_1^*(\delta\bar{\alpha}) \cup p_2^*(\beta) + 0$$

We get that the square commutes.