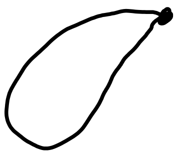
Exercise 1. *Every simply connected manifold is orientable.*

M a simply connected manif of dim n

For every closed curve $\alpha : [0,1] \to M$
$\alpha(0) = \alpha(1)$

there is a homotopy

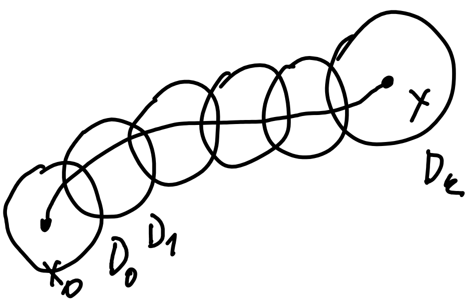


$h\ [0,1] \times [0,1] \to M$
$h(0,t) = \alpha(t)$
$h(1,t) = \alpha(0)$
$h(\tau,0) = h(\tau,1) = \alpha(0)$

Take $x_0 \in M$ fixed $\quad H_n(M, M \setminus x_0) \cong \mathbb{Z}$

we can choose a generator $\underline{\mu_{x_0}}$.

all other orientations $\mu_x \quad x \in M$.



$\alpha\ [0,1] \to M$
$\alpha(0) = x_0$
$\alpha(1) = x$

$\exists$ finite number of closed disks covering the curve

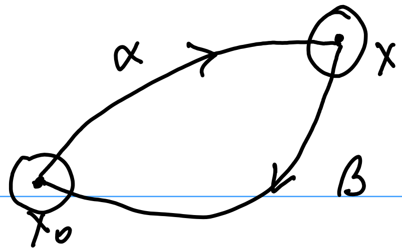
$D_0 \quad H_n(M, M \setminus D_0) \xrightarrow{\cong} H_n(M, M \setminus x_0)$
$\mu_{D_0} \qquad\qquad \mu_{x_0}$

We get μ_y for all $y \in \operatorname{int} D_0$

we get μ_y for some $y \in D_1$
μ_{D_1} - - - - orient. for whole disk

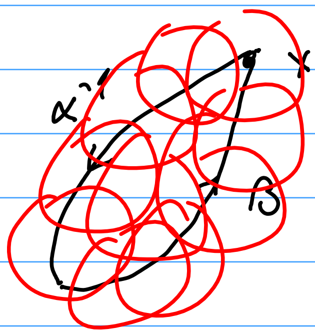
in this way we get orientation for all disks $A = \bigcup_{i=0}^{k} D_i$ is compact $\quad \mu_A \in H_n(M, M \setminus A)$

it gives orient for μ_x, $x = \alpha(1) \in A$



We have prove that we get the same orientation for all curves.

If not



$\gamma : [0,1] \to M$

$\gamma(0) = \gamma(1) = x$

$\mu(\gamma(0)) \neq \mu(\gamma(1))$

$$h : \overset{\tau}{[0,1]} \times \overset{t}{[0,1]} \longrightarrow M$$

$\operatorname{im} h \subseteq M$ is compact

$$h(0,t) = \gamma(t)$$
$$h(1,t) = x = \gamma(0) = \gamma(1)$$
$$h(\tau,0) = h(\tau,1) = x$$

we can get similar covering by disks in the way as for the curve

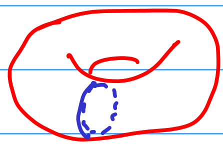
we get compatible orientations

$$A = \bigcup D_i \qquad \mu_A \in H(M, M \setminus A) \to H_n(M, M-x)$$

$$\mu(\gamma(0)) \xleftarrow{\quad} \mu_A \xrightarrow{\quad} \mu(\gamma(1))$$

$$\Rightarrow \quad \mu(\gamma(0)) = \mu(\gamma(1))$$

$T = S^1 \times S^1$
is oriented



Exercise 2. *Prove the modified Poincaré duality with cup product.*

PD: $D: H^k(M;R) \xrightarrow{\cong} H_{n-k}(M;R)$

$\varphi \longmapsto M \cap \varphi$

modified PD: R is a field

mPD $H^k(M;R) \otimes H^{n-k}(M;R) \longrightarrow R$

$\alpha \otimes \beta \longmapsto (\alpha \cup \beta)[M]$

this is a regular bilinear form

PD $\Rightarrow$ mPD $\psi([M] \cap \varphi) = (\varphi \cup \psi)[M]$

$H^{n-k}(M) \xrightarrow[? \cong]{} \mathrm{Hom}(H^k(M;R), R)$

$H^{n-k}(M;R) \xrightarrow{h} \mathrm{Hom}(H_{n-k}(M);R)$

$\alpha(\partial d) = 0$ $[\alpha]$ α is cocycle

$[c]$ c is a cycle $\partial c = 0$

$\alpha \longmapsto h([\alpha])[c] = \alpha(c)$

This is well defined.

$$\begin{tikzcd} H^{n-k}(M;R) \arrow[r, "h"] & \mathrm{Hom}(H_{n-k}(M);R) \arrow[r, "D^*", "\cong"'] & \mathrm{Hom}(H^k(M;R);R) \end{tikzcd}$$

$\psi \in H^{n-k}(M;R)$, $\varphi \in H^k(M;R)$, D

If R is a field also $\cong$

$$\left(D^* h(\psi)\right)(\varphi) = h(\psi) D(\varphi) = h(\psi)([M] \cap \varphi)$$

$$= \psi([M] \cap \varphi) = (\varphi \cup \psi)[M] \quad \text{this is our bilinear form}$$

Exercise 3. *Compute the cohomology ring of $\mathbb{R}P^n$ with $\mathbb{Z}/2$ coefficients.*

In the lecture $H^*(\mathbb{C}P^n;\mathbb{Z}) \cong \mathbb{Z}[w]/\langle w^n \rangle$

$w \in H^2(\mathbb{C}P^n;\mathbb{Z}) \cong \mathbb{Z}$

In the same way we compute

$$H^*(\mathbb{R}P^n;\mathbb{Z}/2) \cong \mathbb{Z}/2[\alpha]/\langle \alpha^{n+1} \rangle$$

$$\alpha \in H^1(\mathbb{R}P^n;\mathbb{Z}/2) \cong \mathbb{Z}/2$$

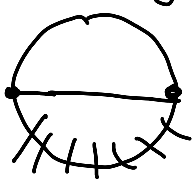
We use: $H^{n-k}(\mathbb{R}P^n;\mathbb{Z}/2) \otimes H^k(\mathbb{R}P^n;\mathbb{Z}/2) \to \mathbb{Z}/2$

is regular.

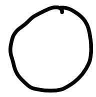
All manifolds are oriented w.r.t. $\mathbb{Z}/2$.

+ induction:

S^1



$\mathbb{R}P^1$



$\mathbb{R}P^1 \cong S^1$

$$H^*(\mathbb{R}P^1;\mathbb{Z}/2) \cong H^*(S^1;\mathbb{Z}/2) \cong \mathbb{Z}/2[\alpha]/\alpha^2=0$$

Suppose it is true for $\mathbb{R}P^{n-1}$

H^1 ... α

H^2 --- $\alpha^2 = \alpha \cup \alpha$

H^{n-1} ---. $\alpha^{n-1} = \alpha \cup \alpha \cup \dots$

$H^{i \geq n}$ 0

$\mathbb{R}P^{n-1} \xhookrightarrow{j} \mathbb{R}P^n$ gives iso

$$H^i(\mathbb{R}P^{n-1};\mathbb{Z}/2) \xleftarrow[\cong]{j^*} H^i(\mathbb{R}P^n;\mathbb{Z}/2) \ni \beta$$

for $i<n$

$\alpha = j^*(\beta)$

$\alpha \cup \alpha = j^*(\beta \cup \beta)$ $\Rightarrow$ $\beta \cup \beta$ is a generator in $H^2(\mathbb{R}P^n;\mathbb{Z}/2)$

etc.

β^{n-1} is a generator in $H^{n-1}(\mathbb{RP}^{n-1};\mathbb{Z}/2)$

$$H^{n-1}(\mathbb{RP}^n;\mathbb{Z}/2)\otimes H^1(\mathbb{RP}^n;\mathbb{Z}/2)\to\mathbb{Z}/2$$

~~in~~ $$\varphi\otimes\psi\longmapsto(\varphi\cup\psi)[\mathbb{RP}^n]$$

is regular $$\beta^{n-1}\otimes\beta\longmapsto\beta^{n-1}\cup\beta\,[\mathbb{RP}^n]$$

$$\beta^n[\mathbb{RP}^n]\neq 0$$

$$\Leftarrow$$

$$\beta^n\neq 0$$

β^n is a generator in $H^n(\mathbb{RP}^n;\mathbb{Z}/2)$

Exercise 4. *Prove that for connected spaces $\bar{H}^*(X \vee Y;\ R) \cong \bar{H}^*(X;\ R) \oplus \bar{H}^*(Y;\ R)$ as graded rings.*

We already know that as graded groups

$$\bar{H}^*(X \vee Y) \cong \underbrace{\bar{H}^*(X)}\ \oplus\ \underbrace{H^*(Y)}$$

ring structure here is given by cup product

ring structures in $\bar{H}^*(X)$ an $\bar{H}^*(Y)$ are given by $\cup$

A a ring, B a ring

We define a ring structure of $\underline{A \oplus B}$

$A^0 \oplus A^1 \oplus \dots \qquad B = B^0 \oplus B^1 \oplus \dots$

$$A \oplus B = (A^0 \oplus B^0) \oplus (A^1 \oplus B^1) \oplus \dots$$

$a, c \in A \qquad (a, b), (c, d) \in A \oplus B$

$b, d \in B$

$$(a, b) \cdot (c, d) = (a \cdot c,\ b d)$$

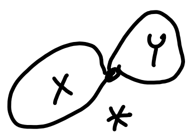
$$(a \cup c,\ b \cup d)$$

$$\begin{tikzcd}
X \arrow[dr, "i_1"] \arrow[dd, "id"'] \arrow[ddrr, bend left=60, "\text{constant map}"] & & Y \arrow[dl, "i_2"'] \arrow[dd, "id"] \arrow[ddll, bend right=80, "\text{const map}"'] \\
& X \vee Y \arrow[dl, "p_1"] \arrow[dr, "p_2"'] & \\
X & & Y
\end{tikzcd}$$

$i_1(x) = x \in X \vee Y$

$p_1(x) = x \qquad p_1(y) = *$

$x \in X \qquad\qquad y \in Y$



$$\bar{H}^n(X) \oplus \bar{H}^n(Y) \rightleftarrows H^n(X \vee Y)$$

$$(a, b) \longmapsto p_1^* a + p_2^* b$$

$$(i_1^* c,\ i_2^* c) \longleftarrow c$$

These are isomorphisms of groups. So it suffices that one of them respects products:

$$(a,b)\cdot(c,d) = (a\cup c,\ b\cup d) \longmapsto$$

$$p_1^*(a\cup c) + p_2^*(b\cup d) = p_1^*(a)\cup p_1^*(c) + p_2^*(b)\cup p_2^*(d)$$

This should be equal to:

$$\left.\begin{array}{l}(a,b) \longmapsto p_1^* a + p_2^* b \\ (c,d) \longmapsto p_1^* c + p_2^* b\end{array}\right\}$$

$$(p_1^* a + p_2^* b)\cup(p_1^* c + p_2^* d) = \underline{p_1^* a\cup p_1^* c} + \underline{p_2^* b\cup p_2^* d}$$

$$+ \underbrace{p_1^* a \cup p_2^* d}_{0} + \underbrace{p_2^* b\cup p_1^* c}_{0}$$

Reason that applying $i_1^*(p_1^* a\cup p_2^* d) =$

$$= \underbrace{i_1^* p_1^*}_{id} a \cup \underbrace{i_1^* p_2^*}_{0} d = a\cup 0 = 0$$

$$i_2^*(p_1^* a\cup p_2^* d) = i_2^* p_1^* a \cup \underbrace{i_2^* p_2^*}_{1} d =$$

$$= 0 \cup d = 0$$

Exercise 5. *Compute the cohomology rings of $\mathbb{C}P^2 \times S^6$ and $\mathbb{C}P^2 \vee S^6$.*

$$H^*(\mathbb{C}P^2) \cong \mathbb{Z}[w]/\langle w^3 \rangle \qquad w \in H^2(\mathbb{C}P^2)$$

$$H^*(S^6) \cong \mathbb{Z}[\alpha]/\langle \alpha^2 \rangle \qquad \alpha \in H^6(S^6)$$

both are free groups

$$H^*(\mathbb{C}P^2 \times S^6) \cong H^*(\mathbb{C}P^2) \otimes H^*(S^6)$$

$$\cong \mathbb{Z}[w]/\langle w^3 \rangle \otimes \mathbb{Z}[\alpha]/\langle \alpha^2 \rangle$$

$$\cong \underline{\mathbb{Z}[w,\alpha]/\langle w^3, \alpha^2 \rangle}$$

From the previous ex. we get

$$H^*(\mathbb{C}P^2 \vee S^6) \cong \tilde{H}^*(\mathbb{C}P^2) \oplus \tilde{H}(S^6) \oplus \underset{\substack{\Psi \\ 1}}{\mathbb{Z}}$$

$$\cong \underline{\mathbb{Z}[w,\alpha]/\langle w^3, \alpha^2, w\alpha \rangle}$$

Exercise 6. *Show that the $\mathbb{C}P^2 \vee S^6$ is not homotopy equivalent to $\mathbb{C}P^3$.*

	$\mathbb{C}P^2 \vee S^6$		$\mathbb{C}P^3$	
H^0	$\mathbb{Z}$	1	$\mathbb{Z}$	1
H^2	$\mathbb{Z}$	$\pm w$	$\mathbb{Z}$	β
H^4	$\mathbb{Z}$	$\pm w^2$	$\mathbb{Z}$	β^2
H^6	$\mathbb{Z}$	α	$\mathbb{Z}$	$\beta^3 = \beta \cup \beta^2$

$\pm w^3 = 0$

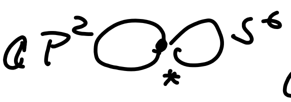
From the graded group structure of H^* of both spaces we cannot distinguish them.

Ring structure

$$H^*(\mathbb{C}P^2 \vee S^6) \cong \mathbb{Z}[\alpha, w] / \langle w^3, \alpha^2, \alpha w \rangle$$
$$H^*(\mathbb{C}P^3) \cong \mathbb{Z}[\beta] / \langle \beta^4 \rangle$$

If both space were hom. eq. then the rings would be isomorphic. But it is not the case.

$\mathbb{C}P^2 \vee S^6$ is not homeo to $\mathbb{C}P^3$



$$\mathbb{C}P^2 \vee S^6 \xrightarrow{f} \mathbb{C}P^3$$

$\underbrace{\mathbb{C}P^2 \vee S^6 \setminus *}_{\text{disconnected}}$ $\qquad f(*)$

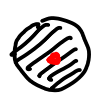
$\mathbb{C}P^3 \setminus f(*)$ is connected

One cannot this argument for hom. equivalence

are hom. eq.

disconn.

$\simeq$



conn.

K cannot be emb. into $\mathbb{R}^3$.

Ex 7 Use Alexander duality to show that a closed nonorientable manifold of dim n cannot be embedded into $\mathbb{R}^{n+1}$.

Klein bottle $H_2(K;\mathbb{Z}) \cong 0$

K is not orientable

$\underline{M}$ nonorientable $\quad$ M can be embedded into $\mathbb{R}^{n+1}$.

$\downarrow$ compactify

M be embedded into S^{n+1}.

Alexander duality $\quad S^{n+1}$

$$H_{n-1}(M;\mathbb{Z}) \cong H^1(S^{n+1} \setminus M;\mathbb{Z})$$

$$H^1(S^{n+1}\setminus M;\mathbb{Z}) \cong \underbrace{\mathrm{Hom}(H_1(S^{n+1}\setminus M),\mathbb{Z})}_{\text{free abelian group}} \oplus \underbrace{\mathrm{Ext}(\underbrace{H_0(S^{n+1}-M)}_{\mathbb{Z}};\mathbb{Z})}_{0}$$

$H^1(S^{n+1}\setminus M;\mathbb{Z})$ is a free ab. group.

$$\underbrace{H_n(M;\mathbb{Z}/2)}_{\text{a.d.}\ \ H^0(S^{n+1}\setminus M;\mathbb{Z}/2)\ \cong\ \mathbb{Z}/2)} \cong \overset{0}{H_n(M;\mathbb{Z})}\otimes\mathbb{Z}/2 \oplus \underbrace{\mathrm{Tor}(\underbrace{H_{n-1}(M;\mathbb{Z})}_{\text{free}},\mathbb{Z}/2)}_{=0}$$