Define the n-th homotopy group of the space X with the base point x_0 as the group of homotopy classes of maps $(I^n, \partial I^n) \to (X, x_0)$ with the operation given by prescription:

$$(\underset{\sim}{f} + \underset{\sim}{g})(t_1, \dots, t_n) = \begin{cases} f(2t_1, t_2, \dots, t_n) & 0 \le t_1 \le \frac{1}{2}, \\ g(2t_1 - 1, t_2, \dots, t_n) & \frac{1}{2} \le t_1 \le 1. \end{cases}$$

Denote it $\pi_n(X, x_0)$.

Exercise 1. *Show the operation on $\pi_n(X, x_0)$ is associative.*

$$\pi_n(X, x_0) = [(I^n, \partial I^n), (X, x_0)]$$

$f+(g+k)$

$f \quad g \quad k$

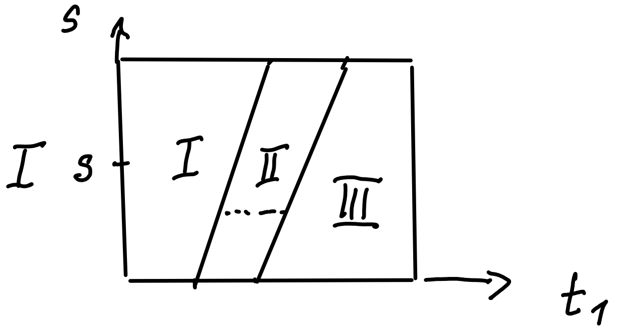
$$\underline{(f+g)+k} \sim \underline{f+(g+k)}$$

$f \quad g \quad k$

$(f+g)+k$

$$I^n \sim I(t_1)$$

$$\underset{s}{I} \times \underset{t_1}{I} \to I$$



I^n

$$h(0, t_1) = [(f+g)+k](t_1)$$

$$h(1, t_1) = [f+(g+k)](t_1)$$

$$h(s, t_1) = \begin{cases} f\left(\frac{4}{1+s} t_1\right) & (s, t_1) \in I \\ g\left(4t_1 - (1+s)\right) & (s, t_1) \in II \\ k\left(\frac{4}{2-s} t_1 - \frac{2+s}{2-s}\right) & (s, t_1) \in III \end{cases}$$

$s = 0$

$s = 1$

$$I: \quad s \in [0,1] \qquad t_1 \in \left[0, \frac{1}{4} + \frac{s}{4}\right]$$

$$II: \quad s \in [0,1] \qquad t_1 \in \left[\frac{1}{4} + \frac{s}{4}, \frac{1}{2} + \frac{s}{4}\right]$$

$$III: \quad s \in [0,1] \qquad t_1 \in \left[\frac{1}{2} + \frac{s}{4}, 1\right]$$

Exercise 2. *Show that the element given by prescription*

$$(-f)(t_1, \ldots, t_n) = f(1 - t_1, t_2, \ldots, t_n)$$

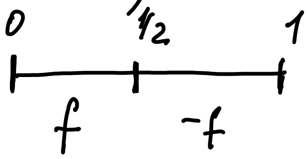
is really the inverse element of f.

$$f : (I^n, \partial I^n) \longrightarrow (X, x_0)$$

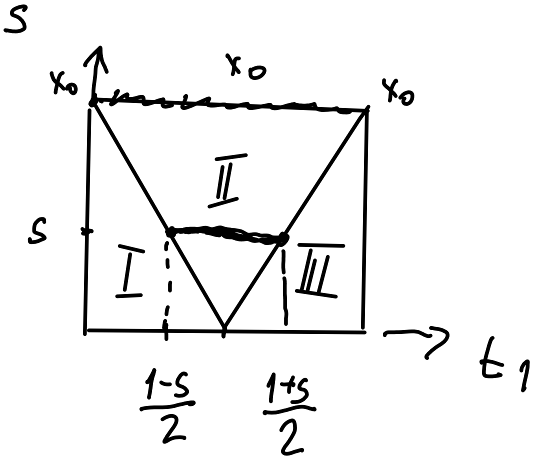
$$(-f) : (I^n, \partial I^n) \longrightarrow (X, x_0)$$

const. into x_0

$f + (-f)$



$\sim$



$$h(s,t_1) = \begin{cases} f(2t_1) & s, t_1 \in \mathrm{I} \\ f\left(\frac{1-s}{2}\right) = (-f)\left(\frac{1+s}{2}\right) & s, t_1 \in \mathrm{II} \\ (-f)(2t_1 - 1) = f(2t_1) & s, t_1 \in \mathrm{III} \end{cases}$$

f $\quad$ $-f$

$0, \frac{1-s}{2}$ $\quad$ $\frac{1+s}{2}, 1$

$$f\left(\frac{1-s}{2}\right) = (-f)\left(\frac{1+s}{2}\right)$$

$f + (-f) \simeq$ const. into x_0

There is a long exact sequence:

$$\cdots \to \pi_{n+1}(X,A,x_0) \xrightarrow{\partial} \pi_n(A,x_0) \xrightarrow{i_*} \pi_n(X,x_0) \xrightarrow{j_*} \pi_n(X,A,x_0) \xrightarrow{\partial} \pi_{n-1}(A,x_0) \to \dots$$

Exercise 3. *Show the exactness of this sequence in* $\pi_n(X,A,x_0)$ *and* $\pi_n(A,x_0)$.

$$\pi_n(X,x_0) \xrightarrow{j_*} \underline{\pi_n(X,A,x_0)} \xrightarrow{\partial} \pi_{n-1}(A,x_0)$$

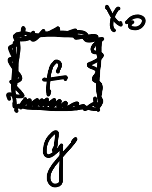
$\partial \circ j_* = 0$ $\quad$ $\operatorname{im} j_* \subseteq \ker \partial$

$[f] \in \pi_n(X,x_0)$ $\quad$ $f:(I^n,\partial I^n) \to (X,x_0)$

$j_* f : (I^n, \partial I^n, J^{n-1}) \to (X,A,x_0)$

$\partial / \partial I^n$ $\quad$ $\partial f = f/\partial I^n \quad (\partial I^n, J^{n-1}) \to (A,x_0)$

$f/\partial I^n = x_0$



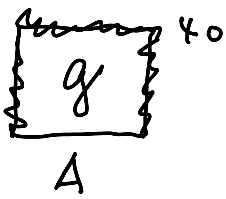
$g = f/_{I^{n-1}} : (I^{n-1}, \partial I^{n-1}) \to (A, x_0)$

g is constant map into x_0

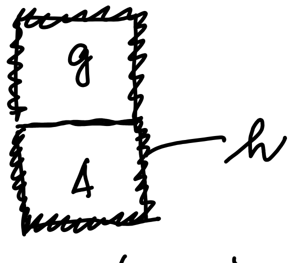
$[g] = 0 \in \pi_{n-1}(A,x_0)$.

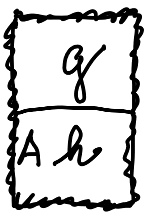
Inclusion $\quad \ker \partial \subseteq \operatorname{im} j_*$

$[g] \in \pi_n(X,A,x_0)$ $\quad$ $g:(I^n,\partial I^n, J^{n-1}) \to (X,A,x_0)$



$g/\partial I^n \sim$ const



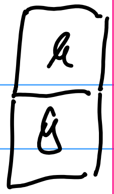


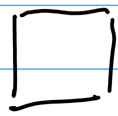
$f:(I^n,\partial I^n) \to (X,x_0)$

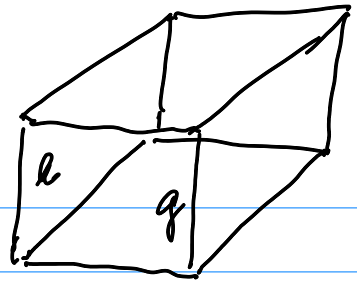
$[f] \in \pi_n(X,x_0)$

$jf : (I^n, \partial I^n, J^{n-1}) \to (X,A,x_0)$

$jf \sim g$







$I^n \times I$

$h \cup x_0$

$$\begin{tikzcd} I^n \times \{0\} \cup \partial I^n \times I \arrow[r, "g"] \arrow[d, hook] & X \\ I^n \times I \arrow[ur, "H"'] & \end{tikzcd}$$

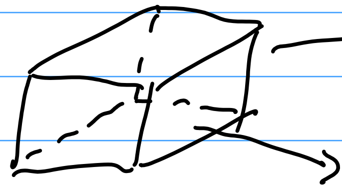
HEP gives extension H

$$\pi_{n+1}(X, A, x_0) \xrightarrow{\partial} \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0)$$

$[h]$

$i_* \circ \partial [h] \sim \text{const}.$

$h : (I^{n+1}, \partial I^{n+1}, J^n) \longrightarrow (X, A, x_0)$



$\longrightarrow h$

$\longrightarrow f = h|_{I^{n-1}} \cdot \pi_n(A, x_0)$

h gives a homotopy between f and const in X

$\text{im}\, \partial \subseteq \ker i_*$

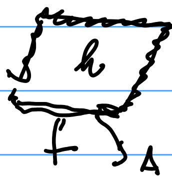
Now we want to prove that

$\ker i_* \subseteq \text{im}\, \partial$

$f \in \ker i_*$ $\qquad f : (I^n, \partial I^n) \longrightarrow (A, x_0)$

$i \circ f : (I^n, \partial I^n) \longrightarrow (X, x_0)$

$i \circ f \sim \text{const} \qquad h : I^n \times I \longrightarrow X$



$h : (I^{n+1}, \partial I^{n+1}, J^n) \longrightarrow (X, A, x_0)$

$\partial h = f$

Exactness in $\pi_n(X, x_0)$ as a homework

Exercise 4. *Show that every fibre bundle is a fibration.*

Fibration

$$\begin{array}{ccc} D^n\times 0 & \longrightarrow & E \\ \downarrow & \nearrow & \downarrow p \\ D^n\times I & \longrightarrow & B \end{array}$$

Fibre bundle $\quad p : E \to B \quad$ F fibre

$B = \bigcup_\alpha U_\alpha$

$$\begin{array}{ccc} p^{-1}(U_\alpha) & \longrightarrow & U_\alpha \times F \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle pr_1} & \\ & U_\alpha & \end{array}$$

(1) Trivial fibre bundle is a fibration

$$\begin{array}{ccc} D^n\times\{0\} & \xrightarrow{f} & B\times F \\ \downarrow & \overset{H}{\nearrow} & \downarrow pr_1 \\ D^n\times I & \xrightarrow{h} & B \end{array}$$

$H(x,t) = \big(h(x,t),\ f(x)\big)$

(2) General fibre bundle

we can replace D^n by I^n

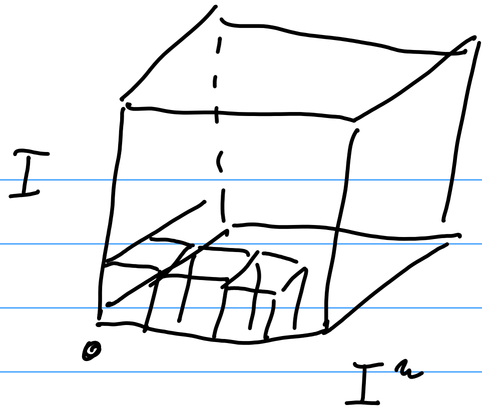
$$\begin{array}{ccc} I^n\times\{0\} & \longrightarrow & E \\ \downarrow & \nearrow & \downarrow \\ I^n\times I & \xrightarrow{h} & B \end{array} \quad \supseteq \quad \begin{array}{c} U_\alpha\times F \\ \downarrow \\ U_\alpha \end{array}$$

compact — we can divide $I^n\times I$ into subcubes

$$C_i \times I_k$$

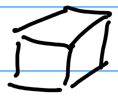
such that

$$h(C_i\times I_k) \subseteq U_\alpha$$

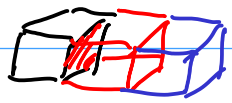


We will construct the lift from one cube to another

$$
\begin{tikzcd}
C_i \times \{0\} \arrow[r] \arrow[d, hook] & U_\alpha \times F \arrow[r, hook] \arrow[d] & E \arrow[d] \\
C_i \times I_k \arrow[r] \arrow[ur, "H"] & U_\alpha \arrow[r, hook] & B
\end{tikzcd}
$$



$$
\begin{tikzcd}
C_{i+1} \times \{0\} \cup \text{part of } \partial C_{i+1} \times I_k \arrow[r] \arrow[d, hook] & U_\beta \times F \arrow[r, hook] \arrow[d] & E \arrow[d] \\
C_{i+1} \times I_k \arrow[r] \arrow[ur, dashed] & U_\beta \arrow[r, hook] & B
\end{tikzcd}
$$



$I^n \times I$

Exercise 5. *Show the structure of the fibre bundle $S^n \xrightarrow{p} \mathbb{R}\mathrm{P}^n$.*

fibre is S^0

$$x \longmapsto \{x, -x\}$$
$$-x \longmapsto \{x, -x\}$$

$[x] \in \mathbb{R}P^n$

$$U_x = \{[x+v],\ v \perp x\}$$

$$p^{-1}(U) \cong U \times S^0$$

$$U \times S^0 \longrightarrow p^{-1}(U) \subseteq S^n$$

$$([x+v], 1) \longmapsto \frac{x+v}{\|x+v\|}$$

$$([x+v], -1) \longmapsto -\frac{x+v}{\|x+v\|}$$

$$U_i = \{[x_0 : x_1 : \dots : x_n],\ x_i \neq 0\}$$

gives the covering of $\mathbb{R}P^n$.

Exercise 6. *Show the structure of the fibre bundle* $S^{2n+1} \xrightarrow{p} \mathbb{C}\mathrm{P}^n$ *with the fibre* S^1.

Fibre is S^1 : $S^{2n+1} \longrightarrow \mathbb{C}P^n$

$$U_i = \{ [z_0 : \dots z_n] \in \mathbb{C}P^n, \ z_i \neq 0 \}$$

$\varphi : U_0 \times S^1 \longrightarrow S^{2n+1}$ homeomorphism

$$\varphi([1 : z_1 : \dots z_n], e^{it}) \longrightarrow \frac{(e^{it}, z_1, z_2 \dots z_n)}{\|(e^{it}, z_1, \dots z_n)\|}$$