Exercise 1. *Show that $(n-1)$-connected compact manifold of dim n is homotopy equivalent to S^n ($n \geq 2$).*

M manifold of dim n, compact

$(n-1)$-connected.

M is homotopy equivalent to S^n.

If M is $(n-1)$-connected

$$\pi_i(M) = 0 \qquad i \leq n-1$$

$$H_0(M) \cong \mathbb{Z} \qquad H_i(M) = 0 \qquad 1 \leq i \leq n-1, \quad i \geq n+1$$

Poincaré duality $\qquad H_n(M) = \mathbb{Z}$

generator fundamental class

$[M] \in H_n(M; \mathbb{Z})$

$$H_0(S^n) \cong \mathbb{Z}, \quad H_i(S^n) = 0 \qquad 1 \leq i \leq n-1, \quad i \geq n+1$$

$$H_n(S^n) = \mathbb{Z}$$

$$\forall i \quad H_i(M) \cong H_i(S^n)$$

Whitehead thm, M, S^n simply connected $\quad n \geq 2$

$f : M \to S^n$ $\qquad$ CW-complexes

$f_* \; H_i(M) \to H_i(S^n)$ iso

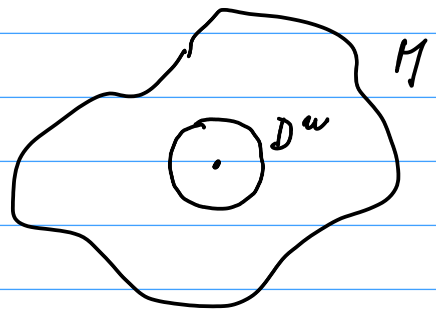
$\Rightarrow$ $f : M \to S^n$ is a homotopy equivalence.

We have to find a suitable map $f : M \to S^n$

$f_* : H_n(M) \to H_n(S^n)$ is an iso

Definition of f

$$D^n \hookrightarrow M$$



$$f : M \longrightarrow D^n / \partial D^n \cong S^n$$

$f(x) = x$ for $x \in \operatorname{int} D^n$
$f(x) = *$ for $x \notin \operatorname{int} D^n$

Evident fact that f is well defined and continuous.

$$f_* \quad H_n(M) \longrightarrow H_n(S^n)$$

$$f_*[M] = [S^n] \Rightarrow f_* \text{ is an iso.}$$

$x_0 \in \operatorname{int} D^n$

$$\begin{tikzcd}
{[M]} \arrow[r, mapsto] \arrow[d, mapsto] & H_n(M) \arrow[r, "f_*", "\cong"'] \arrow[d] & H_n(S^n) \ni [S^n] \arrow[d] \\
\omega_{x_0} \arrow[r, phantom, "\in"] & H_n(M, M \setminus x_0) \arrow[r, "f_*", "\cong"'] & H_n(S^n, S^n \setminus x_0) \ni \omega_{x_0} \text{ generator}
\end{tikzcd}$$

$$H_n(M, M \setminus x_0) \cong \mathbb{Z}$$

$$f_* : H_0(M) \longrightarrow H_0(S^n) \quad \text{is iso}$$
$$[x_0] \longmapsto [x_0]$$

Assumptions of Whitehead thm (homology version) are satisfied $\Rightarrow f : M \to S^n$ is an iso.

Remark. *We have got that a compact manifold 3-dim (compact) which is ~~2-connected~~ simply connected is homotopy equivalent to S^3. There is another result: Every 3-dim manifold simply connected compact manifold is homeomorphic to S^3. This latter result (it might seem we are close to proving it) is actually famous Poincaré conjecture, one of Millenium Prize Problems and it was already solved by Grigori Perelman in 2002. Interesting story and interesting mathematician for sure. Perelman declined Fields medal (among other prizes).*

$\dim M = 3$, compact, simply connected

$M \cong S^3$ homeomorphic

Here: M is 1-connected $\Rightarrow$ M is 2-connected

$\pi_1(M) = 0 \Rightarrow H_1(M) = 0$

CW-complex $f : X \to M$ all cells in X are in dim ≥ 2

$f_* \pi_i X \to \pi_i(M)$ weak hom. equiv.

M is hom. equiv to CW

f is a hom. equivalence.

$H^1(M; \mathbb{Z}) \cong H^1(X; \mathbb{Z}) \cong H^1_{CW}(X; \mathbb{Z}) \cong 0$

$X = e^0 \cup e^2 \dots$

We use Poincaré duality

$H^1(M; \mathbb{Z}) = 0 \iff H_2(M; \mathbb{Z}) = 0$

$\Rightarrow \pi_2(M) = 0 \Rightarrow M$ is 2-connected.

Using our previous statement, we have shown

$M \to S^3$ hom. equivalent.

MUCH LESS THAN HOMEOMORPHIC.

Examination terms:

Tuesday Feb 4th 9 a.m.

or/and • Wednesday Feb 17th 1+1 3 p.m.