Exercise 1. *Show that* $(n-1)$*-connected compact manifold of dim n is homotopy equivalent to* S^n *(*$n \geq 2$*).*

Solution. Take manifold M as two parts: disk and its complement. Map the disk to $S^n = D^n/S^{n-1}$, that is the identity on the interior of the disk and the complement goes to one point. We have $f\colon M \to S^n$ and for $i < n$ by Hurewicz theorem (M is $(n-1)$-connected) we get an iso $f_* : H_i(M) \cong H_i(S^n) = 0$. The same iso we obtain for $i > n$ (both H_i are zero).

We conclude our application with this spectacular diagram where all arrows are iso, reasoning goes from the bottom arrows by excision, then with definiton of fundamental class we get the top arrow (our main focus) an iso. (we denote interior of B as B^o)

$$
\begin{array}{ccc}
H_n(M) & \longrightarrow & H_n(S^n) \\
\downarrow & & \downarrow \\
H_n(M, M - (B^n)^o) & \longrightarrow & H_n(S^n, S^n - (B^n)^o) \\
\downarrow & & \downarrow \\
H_n(disk, \partial disk) & \xrightarrow[\text{id}]{} & H_n(disk, \partial disk)
\end{array}
$$

□

Remark. *We have got that a compact manifold 3-dim (compact) which is 2-connected is homotopy equivalent to* S^3*. There is another result: Every 3-dim manifold simply connected compact manifold is homeomorphic to* S^3*. This latter result (it might seem we are close to proving it) is actually famous Poincaré conjecture, one of Millenium Prize Problems and it was already solved by Grigori Perelman in 2002. Interesting story and interesting mathematician for sure. Perelman declined Fields medal (among other prizes).*