Parametric Curves and Parametric Surfaces

Parametric Curve

A parametric curve in $\mathbb{R}^3$ is given by

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

where $a \leq t \leq b$

There is *one parameter*, because a curve is a *one-dimensional* object

There are *three component functions*, because the curve lives in *three-dimensional* space.

Parametric Surface

A parametric surface in $\mathbb{R}^3$ is given by

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$$

where (u, v) lie in a region D of the uv plane.

There are *two parameters*, because a surface is a *two-dimensional* object

There are *three component functions*, because the surface lives in *three-dimensional* space.

You Are Living on a Parametric Surface

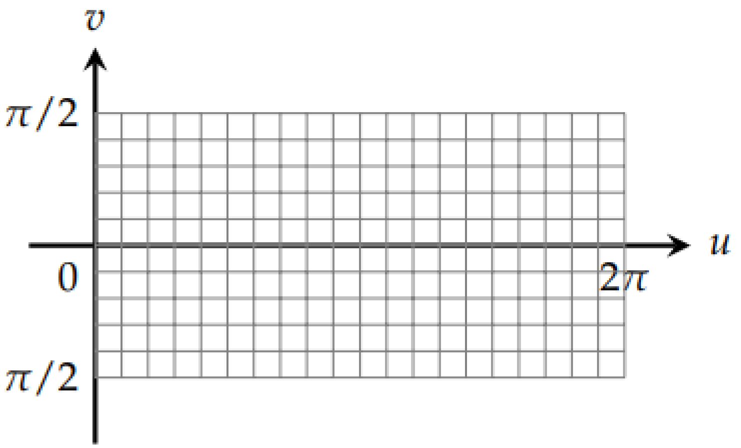
Let u be your longitude (in radians, for this course)

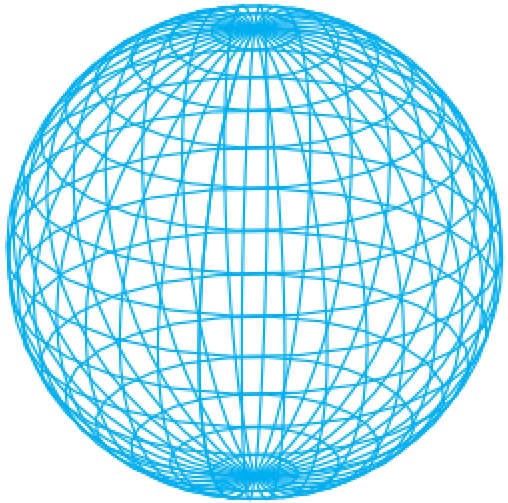
Let v be your latitude (in radians)

Let R be the radius of the Earth

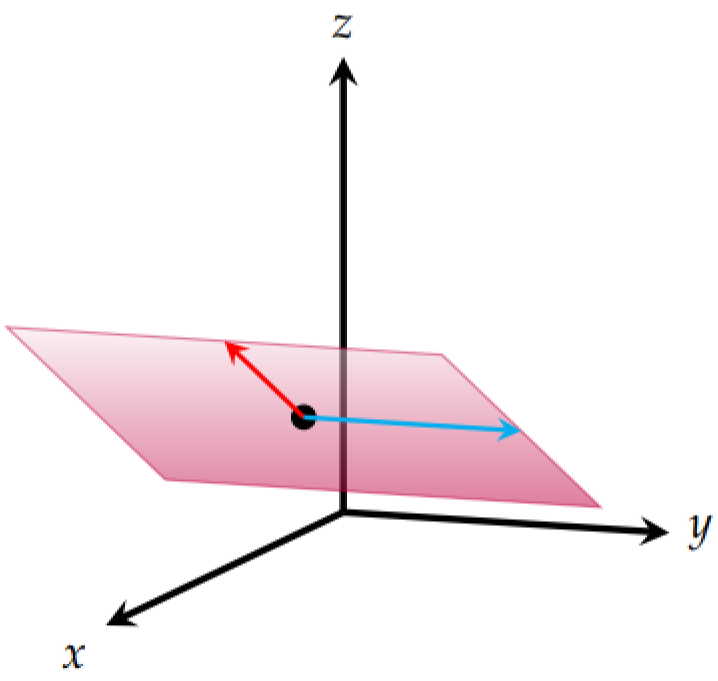
Your position is

$$\mathbf{r}(u,v) = R\cos(v)\cos(u)\mathbf{i} + R\cos(v)\sin(u)\mathbf{j} + R\sin(v)\mathbf{k}$$





More Parameterized Surfaces: Planes



Problem: Find a parametric representation for the plane through $\langle 1,0,1\rangle$ that contains the vectors $\langle 2,0,1\rangle$ and $\langle 0,2,0\rangle$

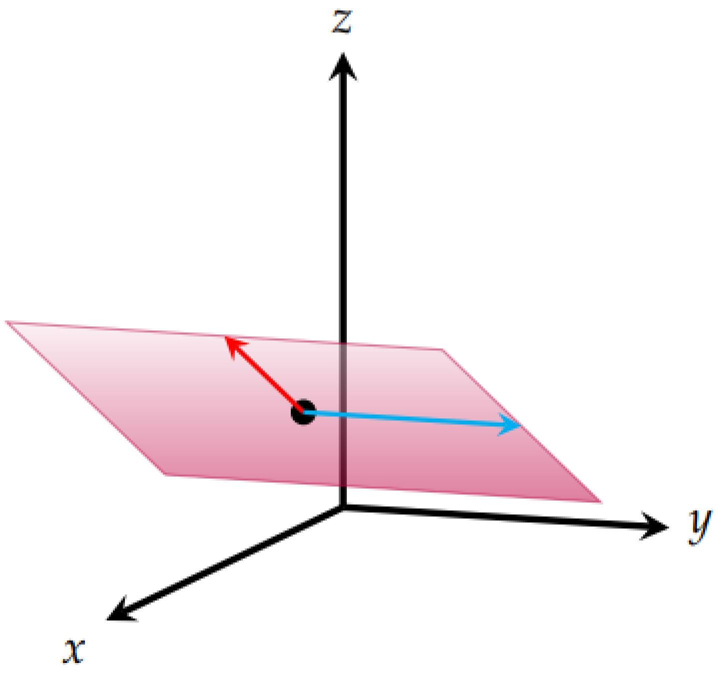
Solution: Let $\mathbf{r}_0 = \langle 1,0,1\rangle$. Any point in the plane is given by

$$\mathbf{r}(s,t) = \langle 1,0,1\rangle + s\langle 2,0,1\rangle + t\langle 0,2,0\rangle$$

Now you try it:

Find a parametric representation for the plane through the point $(0,-1,5)$ that contains the vectors $\langle 2,1,4\rangle$ and $\langle -3,2,5\rangle$.

More Parameterized Surfaces: Planes



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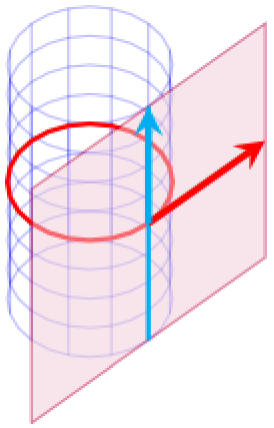


The Tangent Vectors $\mathbf{r}_u$ and $\mathbf{r}_v$

Suppose

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}, \quad (u,v) \in D$$

is a parameterized surface.

At a point $P_0 = \mathbf{r}(u_0, v_0)$, the vectors

$$\mathbf{r}_u(u_0,v_0) = \frac{\partial x}{\partial u}(u_0,v_0)\mathbf{i} + \frac{\partial y}{\partial u}(u_0,v_0)\mathbf{j} + \frac{\partial z}{\partial u}(u_0,v_0)\mathbf{k}$$

$$\mathbf{r}_v(u_0,v_0) = \frac{\partial x}{\partial v}(u_0,v_0)\mathbf{i} + \frac{\partial y}{\partial v}(u_0,v_0)\mathbf{j} + \frac{\partial z}{\partial v}(u_0,v_0)\mathbf{k}$$

are *both* tangent to the surface.

The Tangent Plane

$$\mathbf{r}_u(u_0, v_0) = \frac{\partial x}{\partial u}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial u}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial u}(u_0, v_0)\mathbf{k}$$

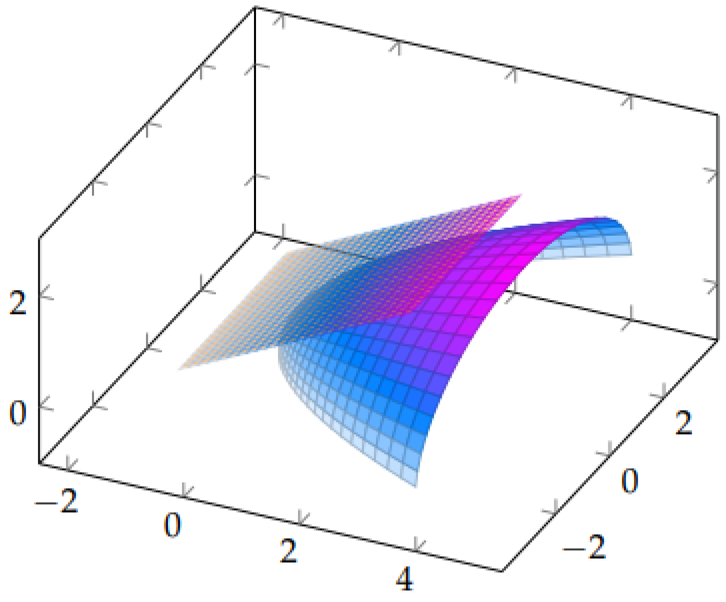
$$\mathbf{r}_v(u_0, v_0) = \frac{\partial x}{\partial v}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial v}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial v}(u_0, v_0)\mathbf{k}$$

The *tangent plane* to a parameterized surface at $P_0 = \mathbf{r}(u_0, v_0)$ is the plane passing through P_0 and perpendicular to $\mathbf{r}_u(u_0, v_0) \times \mathbf{r}_v(u_0, v_0)$.

Find the equation of the tangent plane to the surface

$$\mathbf{r}(u, v) = u^2\mathbf{i} + 2u \sin v\mathbf{j} + u \cos v\mathbf{k}$$

at $u = 1, v = 0$.



$$\mathbf{r}(u,v) = \langle u^2, 2u\sin v, u\cos v\rangle$$
$$\mathbf{r}_u(u,v) = \langle 2u, 2\sin v, \cos v\rangle$$
$$\mathbf{r}_v(u,v) = \langle 0, 2u\cos v, -u\sin v\rangle$$

$$\mathbf{r}(1,0) = \langle 1,0,1\rangle$$
$$\mathbf{r}_u(1,0) = \langle 2,0,1\rangle$$
$$\mathbf{r}_v(1,0) = \langle 0,2,0\rangle$$

The normal to the plane is

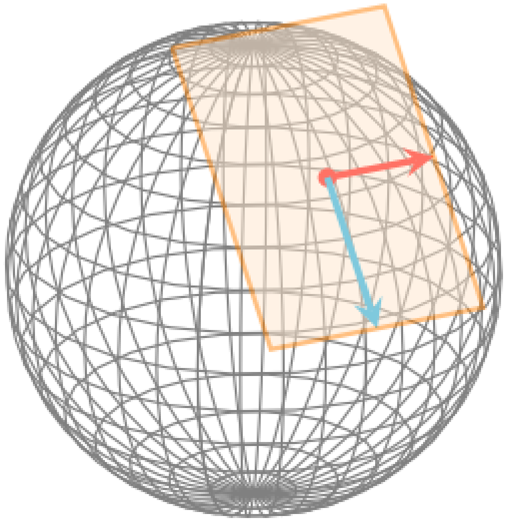
$$\mathbf{r}_u \times \mathbf{r}_v = \langle -1,0,2\rangle$$

so the equation of the plane is

$$(-1)(x-1) + 2(z-1) = 0$$

The tangent plane to the surface at $(1,0,1)$ is parameterized by

$$\langle 1+2s, 2t, 1+s\rangle$$



$$\mathbf{r}(u,v) = \sin(v)\cos(u)\mathbf{i} + \sin(v)\sin(u)\mathbf{j} + \cos(v)\mathbf{k}$$

$$0 \le u \le 2\pi,\ 0 \le v \le \pi$$

$$\mathbf{r}_u = -\sin(v)\sin(u)\mathbf{i} + \sin(v)\cos(u)\mathbf{j}$$
$$\mathbf{r}_v = \cos(v)\cos(u)\mathbf{i} + \cos(v)\sin(u)\mathbf{j} - \sin(v)\mathbf{k}$$

Find the tangent plane to the sphere at $(u,v) = (\pi/4, \pi/4)$

$$\mathbf{r}(\pi/4,\pi/4) = \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} + \frac{\sqrt{2}}{2}\mathbf{k}$$
$$\mathbf{r}_u(\pi/4,\pi/4) = -\frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$$
$$\mathbf{r}_v(\pi/4,\pi/4) = \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} - \frac{\sqrt{2}}{2}\mathbf{k}$$

$$\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v = -\frac{1}{2}\left(\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} + \mathbf{k}\right)$$
$$0 = \frac{1}{\sqrt{2}}(x - \frac{1}{2}) + \frac{1}{\sqrt{2}}(y - \frac{1}{2}) + (z - \frac{\sqrt{2}}{2})$$